

MINIMAL SPECTRAL FUNCTIONS OF AN ORDINARY DIFFERENTIAL OPERATOR

VADIM MOGILEVSKII

ABSTRACT. Let $l[y]$ be a formally selfadjoint differential expression of an even order on the interval $[0, b)$ ($b \leq \infty$) and let L_0 be the corresponding minimal operator. By using the concept of a decomposing boundary triplet we consider the boundary problem formed by the equation $l[y] - \lambda y = f$ ($f \in L_2[0, b)$) and the Nevanlinna λ -depending boundary conditions with constant values at the regular endpoint 0. For such a problem we introduce the concept of the m -function, which in the case of selfadjoint decomposing boundary conditions coincides with the classical characteristic (Titchmarsh-Weyl) function. Our method allows one to describe all minimal spectral functions of the boundary problem, i.e., all spectral functions of the minimally possible dimension. We also improve (in the case of intermediate deficiency indices $n_{\pm}(L_0)$ and not decomposing boundary conditions) the known estimate of the spectral multiplicity of the (exit space) selfadjoint extension $\tilde{A} \supset L_0$. The results of the paper are obtained for expressions $l[y]$ with operator valued coefficients and arbitrary (equal or unequal) deficiency indices $n_{\pm}(L_0)$.

1. INTRODUCTION

The main objects of the paper are differential operators generated by a formally selfadjoint differential expression $l[y]$ of an even order $2n$ on an interval $\Delta = [0, b)$ ($b \leq \infty$). We consider the expression $l[y]$ with operator valued coefficients and arbitrary (possibly unequal) deficiency indices, but in order to simplify presentation of the main results assume that

$$l[y] = \sum_{k=1}^n (-1)^k (p_{n-k} y^{(k)})^{(k)} + p_n y \quad (1.1)$$

is a scalar expression with real-valued coefficients $p_k(t)$ ($t \in \Delta$) [20]. Denote by L_0 and $L(= L_0^*)$ minimal and maximal operators respectively generated by the expression (1.1) in the Hilbert space $\mathfrak{H} := L_2(\Delta)$ and let \mathcal{D} be the domain of L . As is known L_0 is a symmetric operator with equal deficiency indices $m = n_{\pm}(L_0)$ and $n \leq m \leq 2n$. Denote also by $n_b := m - n$ the defect number of the expression (1.1) at the point b [17].

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In the present paper we develop an approach based on the concept of a decomposing boundary triplet for a differential operator [17, 18, 19]. Recall that according to [17] a decomposing boundary triplet for L is a boundary triplet $\Pi = \{\mathbb{C}^n \oplus \mathbb{C}^{n_b}, \Gamma_0, \Gamma_1\}$ in the sense of [9] with the boundary operators $\Gamma_j : \mathcal{D} \rightarrow \mathbb{C}^n \oplus \mathbb{C}^{n_b}$, $j \in \{0, 1\}$ of the special form

$$\Gamma_0 y = \{y^{(2)}(0), \Gamma'_0 y\} (\in \mathbb{C}^n \oplus \mathbb{C}^{n_b}), \quad \Gamma_1 y = \{-y^{(1)}(0), \Gamma'_1 y\} (\in \mathbb{C}^n \oplus \mathbb{C}^{n_b}). \quad (1.2)$$

Here $y^{(j)}(0)$ are vectors of quasi-derivatives (2.27) at the point 0 and $\Gamma'_j y (\in \mathbb{C}^{n_b})$, $j \in \{0, 1\}$ are vectors of boundary values of a function $y \in \mathcal{D}$ at the singular endpoint b .

Next assume that $\mathcal{P} = \{C_0(\lambda), C_1(\lambda)\}$ ($\lambda \in \mathbb{C} \setminus \mathbb{R}$) is a Nevanlinna operator pair defined by the block representations

$$C_0(\lambda) = (\hat{C}_0 \ C'_0(\lambda)) : \mathbb{C}^n \oplus \mathbb{C}^{n_b} \rightarrow \mathbb{C}^m, \quad C_1(\lambda) = (\hat{C}_1 \ C'_1(\lambda)) : \mathbb{C}^n \oplus \mathbb{C}^{n_b} \rightarrow \mathbb{C}^m \quad (1.3)$$

with the constant entries \hat{C}_0 , \hat{C}_1 and let $\tau = \tau(\lambda) := \{\{h, h'\} : C_0(\lambda)h + C_1(\lambda)h' = 0\}$ be the corresponding Nevanlinna family of linear relations. Denote by $\hat{\mathcal{K}}$ the range of the operator $\hat{C} = (\hat{C}_0 \ \hat{C}_1)$ and let

$$\hat{n} = \dim \hat{\mathcal{K}} = \text{rank}(\hat{C}_0 \ \hat{C}_1), \quad n' = m - \hat{n}.$$

Then $n \leq \hat{n} \leq m$ and the operator pair (1.3) admits the block representation

$$C_0(\lambda) = \begin{pmatrix} N_0 & C'_{01}(\lambda) \\ 0 & C'_{02}(\lambda) \end{pmatrix} : \mathbb{C}^n \oplus \mathbb{C}^{n_b} \rightarrow \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp, \quad (1.4)$$

$$C_1(\lambda) = \begin{pmatrix} N_1 & C'_{11}(\lambda) \\ 0 & C'_{12}(\lambda) \end{pmatrix} : \mathbb{C}^n \oplus \mathbb{C}^{n_b} \rightarrow \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp, \quad (1.5)$$

where N_j are $(\hat{n} \times n)$ -matrices with $\text{rk}(N_0 \ N_1) = \hat{n}$ and $C'_{j1}(\lambda)$, $C'_{j2}(\lambda)$ ($j \in \{0, 1\}$) are respectively $(\hat{n} \times n_b)$ and $(n' \times n_b)$ -matrix functions. By using the boundary operators (1.2) consider the boundary problem

$$l[y] - \lambda y = f \quad (1.6)$$

$$(*) \quad C_0(\lambda)\Gamma_0 y - C_1(\lambda)\Gamma_1 y = 0.$$

It follows from (1.4), (1.5) that the boundary condition $(*)$ can be written as two equalities

$$N_0 y^{(2)}(0) + N_1 y^{(1)}(0) + C'_{01}(\lambda)\Gamma'_0 y - C'_{11}(\lambda)\Gamma'_1 y = 0, \quad (1.7)$$

$$C'_{02}(\lambda)\Gamma'_0 y - C'_{12}(\lambda)\Gamma'_1 y = 0, \quad (1.8)$$

which define in fact m linearly independent boundary conditions in the sense of [7].

The problem (1.6)-(1.8) is a particular case of a general Nevanlinna type boundary problem and hence it generates a generalized resolvent $R(\lambda) = R_\tau(\lambda)$ and the corresponding spectral function $F(t) = F_\tau(t)$ of the operator L_0 [19]. Moreover each selfadjoint boundary problem is given by the boundary condition $(*)$ with a constant-valued Nevanlinna pair $\mathcal{P} = \{C_0, C_1\}$, which implies that each canonical resolvent of the operator L_0 is generated by the boundary problem (1.6)-(1.8) with

$C'_{j1}(\lambda) \equiv C'_{j1}$ and $C'_{j2}(\lambda) \equiv C'_{j2}$, $j \in \{0, 1\}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Observe also that the problem (1.6)-(1.8) contains as a particular case a decomposing boundary problem. Namely if (and only if) the equality $\hat{n} = n$ is satisfied, then $C'_{01}(\lambda) = C'_{11}(\lambda) = 0$ and the boundary conditions (1.7), (1.8) becomes decomposing.

Next assume that $M(\cdot)$ is the Weyl function of the decomposing boundary triplet (1.2) in the sense of [4] and let

$$M(\lambda) = \begin{pmatrix} m(\lambda) & M_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : \mathbb{C}^n \oplus \mathbb{C}^{n_b} \rightarrow \mathbb{C}^n \oplus \mathbb{C}^{n_b}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \quad (1.9)$$

be the block representation of $M(\lambda)$. Moreover let $\Omega(\lambda) = \Omega_\tau(\lambda)$ be the Shtraus characteristic matrix of the generalized resolvent $R(\lambda) = R_\tau(\lambda)$ [24]. Then $\Omega_\tau(\lambda)$ is defined immediately in terms of a Nevanlinna boundary parameter τ by the equalities

$$\tilde{\Omega}_\tau(\lambda) = \begin{pmatrix} M(\lambda) - M(\lambda)(\tau(\lambda) + M(\lambda))^{-1}M(\lambda) & -\frac{1}{2}I + M(\lambda)(\tau(\lambda) + M(\lambda))^{-1} \\ -\frac{1}{2}I + (\tau(\lambda) + M(\lambda))^{-1}M(\lambda) & -(\tau(\lambda) + M(\lambda))^{-1} \end{pmatrix}, \quad (1.10)$$

$$\Omega_\tau(\lambda) = P_{\mathbb{C}^n \oplus \mathbb{C}^n} \tilde{\Omega}_\tau(\lambda) \upharpoonright \mathbb{C}^n \oplus \mathbb{C}^n, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

(see [19]). For a given operator pair (1.4), (1.5) consider also the operator function

$$\Omega_{\tau, W'}(\lambda) = (W')^{-1} \Omega_\tau(\lambda) (W')^{-1*}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (1.11)$$

where $W' = \begin{pmatrix} -N_0^* & * \\ N_1^* & * \end{pmatrix}$ is an invertible $(2n \times 2n)$ -matrix (the form of the entries $*$ does not matter). We show in the paper that the operator function (1.11) is of the form

$$\Omega_{\tau, W'}(\lambda) = \begin{pmatrix} m_{\mathcal{P}}(\lambda) & C^* \\ C & 0 \end{pmatrix} : \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp \rightarrow \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (1.12)$$

where C is a constant operator. The equality (1.12) generates the uniformly strict Nevanlinna operator function $m_{\mathcal{P}}(\cdot)$, which we call an m -function of the boundary problem (1.6)-(1.8). This function can be also explicitly defined in terms of the boundary conditions (1.7), (1.8) (see Theorem 3.11, 3)). Moreover in the case of selfadjoint decomposing boundary conditions the function $m_{\mathcal{P}}(\cdot)$ coincides with the classical characteristic (Titchmarsh-Weyl) function [20].

It turns out that the characteristic matrix $\Omega_\tau(\cdot)$ and the m -function $m_{\mathcal{P}}(\cdot)$ are connected by

$$m_{\mathcal{P}}(\lambda) = \hat{N}^* \Omega_\tau(\lambda) \hat{N} + \hat{C}, \quad \hat{C} = \hat{C}^*,$$

where \hat{N} is the right inverse operator for $N' = (-N_0 \ N_1)$. This implies that $m_{\mathcal{P}}(\cdot)$ is the uniformly strict part of the Nevanlinna function $\Omega_\tau(\cdot)$ and the function $\Omega_\tau(\cdot)$ is uniformly strict if and only if $m = \hat{n} = 2n$ and the $(2n \times 2n)$ -matrix $(N_0 \ N_1)$ is invertible.

In the final part of the paper we consider some questions of the eigenfunction expansion. Namely let $\varphi(t, \lambda) = (\varphi_1(t, \lambda) \ \varphi_2(t, \lambda) \ \dots \ \varphi_d(t, \lambda))$ be a system of $d = d_\varphi$ linearly independent solutions of the equation $l[y] - \lambda y = 0$ with the constant initial data $\varphi^{(j)}(0, \lambda) \equiv \varphi_j$, $j \in \{0, 1\}$. Recall that a $(d \times d)$ -matrix distribution

$\Sigma(s) = \Sigma_{\tau, \varphi}(s)$ ($s \in \mathbb{R}$) is called a spectral function of the boundary problem (1.6)-(1.8) corresponding to the solution $\varphi(\cdot, \lambda)$ if for each function $f \in \mathfrak{H}$ with compact support the Fourier transform

$$g_f(s) = \int_0^b \varphi^\top(t, s) f(t) dt.$$

satisfies the equality

$$((F_\tau(\beta) - F_\tau(\alpha))f, f)_{\mathfrak{H}} = \int_{[\alpha, \beta]} (d\Sigma_{\tau, \varphi}(s) g_f(s), g_f(s)), \quad [\alpha, \beta] \subset \mathbb{R}.$$

(here $F_\tau(\cdot)$ is the spectral function of L_0). As is known [7, 20, 24] in the case $d_\varphi = 2n$ there exists a unique spectral function $\Sigma_{\tau, \varphi}(\cdot)$ of the problem (1.6)-(1.8). At the same time for simplification of calculations it is important to make d_φ as small as possible [7, ch. 13.5]. Therefore the natural problem seems to be a description of all spectral functions $\Sigma_{\tau, \varphi}(\cdot)$ with the minimally possible value of d_φ (we denote this value by d_{\min} and we call the corresponding spectral function minimal). It turns out that the complete solution of this problem is based on the introduced concept of the m -function $m_{\mathcal{P}}(\cdot)$. Namely the following theorem holds.

Theorem 1.1. *Let $\mathcal{P} = \{C_0(\lambda), C_1(\lambda)\}$ be a Nevanlinna pair (1.4), (1.5) and let $\varphi_N(t, \lambda) = (\varphi_1(t, \lambda) \ \varphi_2(t, \lambda) \ \dots \ \varphi_{\hat{n}}(t, \lambda))$ be the \hat{n} -component linearly independent solution of the equation $l[y] - \lambda y = 0$ with the initial data $\varphi_N^{(1)}(0, \lambda) = -N_0^*$, $\varphi_N^{(2)}(0, \lambda) = N_1^*$. Then:*

1) *there exists the unique $(\hat{n} \times \hat{n})$ -spectral function $\Sigma_{\mathcal{P}, N}(s)$ of the problem (1.6)-(1.8) corresponding to $\varphi_N(\cdot, \lambda)$ and this function is calculated by means of the Stieltjes formula (4.17) for the m -function $m_{\mathcal{P}}(\cdot)$;*

2) *$d_{\min} = \hat{n}$ and the set of all minimal spectral functions $\Sigma_{\min}(\cdot)$ is given by*

$$\Sigma_{\min}(s) = X^* \Sigma_{\mathcal{P}, N}(s) X,$$

where X is an invertible $(\hat{n} \times \hat{n})$ -matrix.

Moreover we show that for a fixed pair $N = (N_0 \ N_1)$ the set of all spectral functions $\Sigma_{\mathcal{P}, N}(s)$ is parameterized by the Stieltjes formula (4.17) and the following equality

$$m_{\mathcal{P}}(\lambda) = T_{N,0}(\lambda) + T_N(\lambda)(C_0(\lambda) - C_1(\lambda)M(\lambda))^{-1}C_1(\lambda)T_N^*(\bar{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (1.13)$$

which is similar to the known Krein formula for resolvents (see for instance [4]). In formula (1.13) $M(\lambda)$ is the Weyl function (1.9) and $T_{N,0}(\lambda)$, $T_N(\lambda)$ are the matrix functions defined by means of $M(\lambda)$ and the pair N . The role of a parameter in (1.13) is played by a Nevanlinna pair $\mathcal{P} = \{C_0(\lambda), C_1(\lambda)\}$ given by (1.4), (1.5) with fixed N_0 , N_1 and all possible $C'_{ij}(\lambda)$. Note in this connection that for a decomposing boundary problem formula (1.13) leads to similar one from our paper [16]. Moreover (1.13) implies the known description of all Titchmarsh - Weyl functions $m(\cdot)$ obtained for quasi-regular expressions $l[y]$ by Fulton [8] and Khol'kin [10, 22] (we are going to touch upon these questions elsewhere).

Finally by using Theorem 1.1 we prove the inequality $sm(\tilde{A}) \leq \hat{n}$, where $sm(\tilde{A})$ is the spectral multiplicity of the (exit space) selfadjoint extension $\tilde{A} \supset L_0$ given by the boundary conditions (1.7), (1.8). This result improves the known estimate $sm(\tilde{A}) \leq m$ implied by simplicity of the operator L_0 . In this connection note that in the case $\Delta = [0, b]$ one can put in (1.7), (1.8) $\Gamma'_0 y = y^{(2)}(b)$, $\Gamma'_1 y = y^{(1)}(b)$, which implies that the multiplicity of each eigenvalue of the canonical extension $\tilde{A} = \tilde{A}^*$ does not exceed $\hat{n} (= \text{rk}(N_0 \ N_1))$. Hence in the case $\Delta = [0, b]$ the estimate $sm(\tilde{A}) \leq \hat{n}$ (for the canonical extension \tilde{A}) is immediate from (1.7), (1.8) and discreteness of spectrum of \tilde{A} . Meanwhile, such an estimate dose not seem to be so obvious in the case of intermediate deficiency indices $n < m < 2n$ and not decomposing boundary conditions.

2. PRELIMINARIES

2.1. Notations. The following notations will be used throughout the paper: \mathfrak{H} , \mathcal{H} denote Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on \mathcal{H}_1 with values in \mathcal{H}_2 ; $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$; $P_{\mathcal{L}}$ is the orthogonal projector in \mathfrak{H} onto the subspace $\mathcal{L} \subset \mathfrak{H}$; \mathbb{C}_+ (\mathbb{C}_-) is the upper (lower) half-plane of the complex plane.

Recall that a closed linear relation from \mathcal{H}_0 to \mathcal{H}_1 is a closed subspace in $\mathcal{H}_0 \oplus \mathcal{H}_1$. The set of all closed linear relations from \mathcal{H}_0 to \mathcal{H}_1 (from \mathcal{H} to \mathcal{H}) will be denoted by $\tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ ($\tilde{\mathcal{C}}(\mathcal{H})$). A closed linear operator T from \mathcal{H}_0 to \mathcal{H}_1 is identified with its graph $\text{gr}T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$.

For a relation $T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\text{Ker}T$ the domain, range and the kernel respectively. Moreover $T^{-1} (\in \tilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0))$ and $T^* (\in \tilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0))$ stands for the inverse and adjoint relations.

In the case $T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we write: $0 \in \rho(T)$ if $\text{Ker}T = \{0\}$ and $\mathcal{R}(T) = \mathcal{H}_1$, or equivalently if $T^{-1} \in [\mathcal{H}_1, \mathcal{H}_0]$; $0 \in \hat{\rho}(T)$ if $\text{Ker}T = \{0\}$ and $\mathcal{R}(T)$ is closed. For a linear relation $T \in \tilde{\mathcal{C}}(\mathcal{H})$ we denote by $\rho(T) = \{\lambda \in \mathbb{C} : 0 \in \rho(T - \lambda)\}$ and $\hat{\rho}(T) = \{\lambda \in \mathbb{C} : 0 \in \hat{\rho}(T - \lambda)\}$ the resolvent set and the set of regular type points of T respectively.

2.2. Holomorphic operator pairs. Recall that a holomorphic operator function $\Phi(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\mathcal{H}]$ is called a Nevanlinna function if $\text{Im} \lambda \cdot \text{Im} \Phi(\lambda) \geq 0$ and $\Phi^*(\lambda) = \Phi(\bar{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover the Nevanlinna function $\Phi(\cdot)$ is said to be uniformly strict if $0 \in \rho(\text{Im} \Phi(\lambda))$.

Next assume that Λ is an open set in \mathbb{C} , $\mathcal{K}, \mathcal{H}_0, \mathcal{H}_1$ are Hilbert spaces and $C_j(\cdot) : \Lambda \rightarrow [\mathcal{H}_j, \mathcal{K}]$, $j \in \{0, 1\}$ is a pair of holomorphic operator functions (briefly a holomorphic pair). In what follows we identify such a pair with a holomorphic operator function

$$C(\lambda) = (C_0(\lambda) \ C_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \rightarrow \mathcal{K}, \quad \lambda \in \Lambda. \quad (2.1)$$

A pair (2.1) will be called admissible if $\mathcal{R}(C(\lambda)) = \mathcal{K}$ for all $\lambda \in \Lambda$. In the sequel all pairs (2.1) are admissible unless otherwise stated.

Definition 2.1. Two holomorphic pairs $C(\cdot) : \Lambda \rightarrow [\mathcal{H}_0 \oplus \mathcal{H}_1, \mathcal{K}]$ and $C'(\cdot) : \Lambda \rightarrow [\mathcal{H}_0 \oplus \mathcal{H}_1, \mathcal{K}']$ are said to be equivalent if $C'(\lambda) = \varphi(\lambda)C(\lambda)$, $\lambda \in \Lambda$ with a holomorphic isomorphism $\varphi(\cdot) : \Lambda \rightarrow [\mathcal{K}, \mathcal{K}']$.

Clearly, the set of all holomorphic pairs (2.1) falls into nonintersecting classes of equivalent pairs. Moreover such a class can be identified with a function $\tau(\cdot) : \Lambda \rightarrow \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ given for all $\lambda \in \Lambda$ by

$$\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{K}\} := \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : C_0(\lambda)h_0 + C_1(\lambda)h_1 = 0\}. \quad (2.2)$$

In what follows we suppose that \mathcal{H}_0 is a Hilbert space, \mathcal{H}_1 is a subspace in \mathcal{H}_0 , $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$ and P_j is the orthoprojector in \mathcal{H}_0 onto \mathcal{H}_j , $j \in \{1, 2\}$. With each linear relation $\theta \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we associate a \times -adjoint linear relation $\theta^\times \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ defined as the set of all $\{k_0, k_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1$ such that

$$(k_1, h_0) - (k_0, h_1) + i(P_2 k_0, P_2 h_0) = 0, \quad \{h_0, h_1\} \in \theta.$$

Clearly, in the case $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ the equality $\theta^\times = \theta^*$ is valid.

Next assume that \mathcal{K}_0 is an auxiliary Hilbert space, \mathcal{K}_1 is a subspace in \mathcal{K}_0 and

$$C(\lambda) = (C_0(\lambda) \ C_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \rightarrow \mathcal{K}_0, \quad \lambda \in \mathbb{C}_+ \quad (2.3)$$

$$D(\lambda) = (D_0(\lambda) \ D_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \rightarrow \mathcal{K}_1, \quad \lambda \in \mathbb{C}_- \quad (2.4)$$

are holomorphic operator pairs with the block-matrix representations

$$C_0(\lambda) = (C_{01}(\lambda) \ C_{02}(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{K}_0, \quad (2.5)$$

$$D_0(\lambda) = (D_{01}(\lambda) \ D_{02}(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{K}_1. \quad (2.6)$$

Definition 2.2. A Nevanlinna collection of holomorphic operator pairs (briefly a Nevanlinna collection) is a totality $\{C(\cdot), D(\cdot)\}$ of holomorphic pairs (2.3), (2.4) satisfying

$$2 \operatorname{Im}(C_1(\lambda)C_{01}^*(\lambda)) + C_{02}(\lambda)C_{02}^*(\lambda) \geq 0, \quad 0 \in \rho(C_0(\lambda) - iC_1(\lambda)P_1), \quad \lambda \in \mathbb{C}_+ \quad (2.7)$$

$$2 \operatorname{Im}(D_1(\lambda)D_{01}^*(\lambda)) + D_{02}(\lambda)D_{02}^*(\lambda) \leq 0, \quad 0 \in \rho(D_{01}(\lambda) + iD_1(\lambda)), \quad \lambda \in \mathbb{C}_- \quad (2.8)$$

$$C_1(\lambda)D_{01}^*(\bar{\lambda}) - C_{01}(\lambda)D_1^*(\bar{\lambda}) + iC_{02}(\lambda)D_{02}^*(\bar{\lambda}) = 0, \quad \lambda \in \mathbb{C}_+. \quad (2.9)$$

A Nevanlinna collection (2.3), (2.4) is said to be constant if $\mathcal{K}_0 = \mathcal{K}_1 =: \mathcal{K}$ and $C_j(\lambda) = D_j(z) \equiv C_j$, $j \in \{0, 1\}$ for all $\lambda \in \mathbb{C}_+$, $z \in \mathbb{C}_-$.

Clearly, a constant Nevanlinna collection can be regarded as an operator pair

$$C = (C_0 \ C_1) : \mathcal{H}_0 \oplus \mathcal{H}_1 \rightarrow \mathcal{K} \quad (2.10)$$

with the block-matrix representation $C_0 = (C_{01} \ C_{02}) : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{K}$ satisfying

$$2 \operatorname{Im}(C_1 C_{01}^*) + C_{02} C_{02}^* = 0, \quad 0 \in \rho(C_0 - iC_1 P_1), \quad 0 \in \rho(C_{01} + iC_1). \quad (2.11)$$

This and Proposition 3.4 in [14] imply that the equality

$$\theta = \{(C_0, C_1); \mathcal{K}\} := \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : C_0 h_0 + C_1 h_1 = 0\} \quad (2.12)$$

define a linear relation $\theta \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ such that $(-\theta)^\times = -\theta$. Moreover a constant Nevanlinna collection exists if and only if $\dim \mathcal{H}_1 = \dim \mathcal{H}_0 (= \dim \mathcal{K})$.

Definition 2.3. A collection $\tau = \{\tau_+, \tau_-\}$ of two functions $\tau_+(\cdot) : \mathbb{C}_+ \rightarrow \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ and $\tau_-(\cdot) : \mathbb{C}_- \rightarrow \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is said to be of the class $\tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ if for all $\lambda \in \mathbb{C}_+$ and $z \in \mathbb{C}_-$ it admits the representation

$$\tau_+(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{K}_0\}, \quad \tau_-(z) = \{(D_0(z), D_1(z)); \mathcal{K}_1\} \quad (2.13)$$

with a Nevanlinna collection $\{C(\cdot), D(\cdot)\}$ (see (2.2)).

A collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to the class $\tilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$ if it admits the representation $\tau_\pm(\lambda) = \{(C_0, C_1); \mathcal{K}\} = \theta$, $\lambda \in \mathbb{C}_\pm$ with a constant Nevanlinna collection (operator pair) (2.10).

It follows from Definition 2.3 that a collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ can be regarded as a collection of two equivalence classes of holomorphic pairs (2.3) and (2.4) satisfying (2.7)–(2.9). Moreover according to [14]

$$\operatorname{Im} \lambda \cdot (2\operatorname{Im}(h_1, h_0) - \|P_2 h_0\|^2) \geq 0, \quad \{h_0, h_1\} \in \tau_\pm(\lambda) \quad (2.14)$$

and $-\tau_+(\lambda) = (-\tau_-(\bar{\lambda}))^\times$, $\lambda \in \mathbb{C}_+$ for any collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$.

Remark 2.4. 1) Clearly a Nevanlinna collection (2.3), (2.4) satisfies the equalities

$$\dim \mathcal{H}_0 = \dim \mathcal{K}_0, \quad \dim \mathcal{H}_1 = \dim \mathcal{K}_1. \quad (2.15)$$

Therefore the representation (2.13) with $\mathcal{K}_0 = \mathcal{K}_1 =: \mathcal{K}$ is possible if and only if $\dim \mathcal{H}_1 = \dim \mathcal{H}_0$, in which case the corresponding Nevanlinna collection (2.3), (2.4) can be regarded as the unique holomorphic operator pair defined on $\mathbb{C}_+ \cup \mathbb{C}_-$.

2) In the case $\mathcal{H}_1 = \mathcal{H}_0 =: \mathcal{H}$ the class $\tilde{R}(\mathcal{H}) := \tilde{R}(\mathcal{H}, \mathcal{H})$ coincides with the known class of Nevanlinna functions $\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \tilde{\mathcal{C}}(\mathcal{H})$ (see for instance [6]) and (2.13) takes the form

$$\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{K}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (2.16)$$

where $C(\lambda) = (C_0(\lambda) \quad C_1(\lambda)) : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{K}$ is a holomorphic Nevanlinna pair. Moreover a constant Nevanlinna pair can be identified by means of (2.10) and (2.12) with a selfadjoint linear relation (operator pair) $\theta = \theta^* \in \tilde{\mathcal{C}}(\mathcal{H})$.

2.3. Boundary triplet and the Weyl function. Let A be a closed densely defined symmetric operator in \mathfrak{H} with the deficiency indices $n_\pm(A) := \dim \mathfrak{N}_\lambda(A)$, $\lambda \in \mathbb{C}_\pm$.

Definition 2.5. [15] A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where \mathcal{H}_0 is a Hilbert space, \mathcal{H}_1 is a subspace in \mathcal{H}_0 and $\Gamma_j : \mathcal{D}(A^*) \rightarrow \mathcal{H}_j$, $j \in \{0, 1\}$ are linear maps, is called a D -boundary triplet (or briefly a D -triplet) for A^* , if the map $\Gamma = (\Gamma_0 \quad \Gamma_1)^\top : \mathcal{D}(A^*) \rightarrow \mathcal{H}_0 \oplus \mathcal{H}_1$ is surjective and the following Green's identity holds

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) + i(P_2 \Gamma_0 f, P_2 \Gamma_0 g), \quad f, g \in \mathcal{D}(A^*)$$

(here as before P_2 is the orthoprojector in \mathcal{H}_0 onto $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$).

As was shown in [15]

$$\dim \mathcal{H}_1 = n_-(A) \leq n_+(A) = \dim \mathcal{H}_0$$

for each D -triplet $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$. Moreover the equalities

$$\mathcal{D}(A_0) := \text{Ker} \Gamma_0 = \{f \in \mathcal{D}(A^*) : \Gamma_0 f = 0\}, \quad A_0 = A^* \upharpoonright \mathcal{D}(A_0) \quad (2.17)$$

define the maximal symmetric extension A_0 of A with $n_-(A_0) = 0$.

It turns out that for every $\lambda \in \mathbb{C}_+$ ($z \in \mathbb{C}_-$) the map $\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(A)$ ($P_1 \Gamma_0 \upharpoonright \mathfrak{N}_z(A)$) is an isomorphism. This makes it possible to introduce the operator functions (γ -fields) $\gamma_+(\cdot) : \mathbb{C}_+ \rightarrow [\mathcal{H}_0, \mathfrak{H}]$, $\gamma_-(\cdot) : \mathbb{C}_- \rightarrow [\mathcal{H}_1, \mathfrak{H}]$ and the Weyl functions $M_+(\cdot) : \mathbb{C}_+ \rightarrow [\mathcal{H}_0, \mathcal{H}_1]$, $M_-(\cdot) : \mathbb{C}_- \rightarrow [\mathcal{H}_1, \mathcal{H}_0]$ by

$$\gamma_+(\lambda) = (\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(A))^{-1}, \quad \lambda \in \mathbb{C}_+; \quad \gamma_-(z) = (P_1 \Gamma_0 \upharpoonright \mathfrak{N}_z(A))^{-1}, \quad z \in \mathbb{C}_-, \quad (2.18)$$

$$\Gamma_1 \upharpoonright \mathfrak{N}_\lambda(A) = M_+(\lambda) \Gamma_0 \upharpoonright \mathfrak{N}_\lambda(A), \quad \lambda \in \mathbb{C}_+, \quad (2.19)$$

$$(\Gamma_1 + iP_2 \Gamma_0) \upharpoonright \mathfrak{N}_z(A) = M_-(z) P_1 \Gamma_0 \upharpoonright \mathfrak{N}_z(A), \quad z \in \mathbb{C}_-. \quad (2.20)$$

According to [15] all functions γ_\pm and M_\pm are holomorphic on their domains and $M_+^*(\lambda) = M_-(\bar{\lambda})$, $\lambda \in \mathbb{C}_+$. Moreover the block matrix representations

$$M_+(\lambda) = (M(\lambda) \ N_+(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1, \quad \lambda \in \mathbb{C}_+ \quad (2.21)$$

$$M_-(z) = (M(z) \ N_-(z))^\top : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2, \quad z \in \mathbb{C}_- \quad (2.22)$$

generate the uniformly strict Nevanlinna function $M(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\mathcal{H}_1]$.

Proposition 2.6. *Let A be a densely defined symmetric operator in \mathfrak{H} , let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a D -triplet for A^* and let $M_+(\cdot)$ be the corresponding Weyl function. Then for each collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ the following equalities hold*

$$s - \lim_{y \rightarrow +\infty} P_1(\tau_+(iy) + M_+(iy))^{-1}/y = 0, \quad (2.23)$$

$$s - \lim_{y \rightarrow +\infty} (M(iy) - M_+(iy)(\tau_+(iy) + M_+(iy))^{-1}M(iy))/y = 0. \quad (2.24)$$

Remark 2.7. If a D -triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ satisfies the relation $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H} (\Leftrightarrow A_0 = A_0^*)$, then it is a boundary triplet. More precisely this means that the collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet (boundary value space) for A^* in the sense of [9]. In this case the relations

$$\gamma(\lambda) = (\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(A))^{-1}, \quad \Gamma_1 \upharpoonright \mathfrak{N}_\lambda(A) = M(\lambda) \Gamma_0 \upharpoonright \mathfrak{N}_\lambda(A), \quad \lambda \in \rho(A_0) \quad (2.25)$$

define the operator functions [4] $\gamma(\cdot) : \rho(A_0) \rightarrow [\mathcal{H}, \mathfrak{H}]$ (the γ -field) and $M(\cdot) : \rho(A_0) \rightarrow [\mathcal{H}]$ (the Weyl function) associated with the operator functions (2.18)–(2.20) by $\gamma(\lambda) = \gamma_\pm(\lambda)$ and $M(\lambda) = M_\pm(\lambda)$, $\lambda \in \mathbb{C}_\pm$. Observe also that for a boundary triplet Π Proposition 2.6 follows from the Π -admissibility criterion obtained in [5, 6].

2.4. Differential operators. Let $\Delta = [0, b)$ ($b \leq \infty$) be an interval on the real axis (in the case $b < \infty$ the point b may or may not belong to Δ), let H be a separable Hilbert space and let

$$l[y] = \sum_{k=1}^n (-1)^k ((p_{n-k} y^{(k)})^{(k)} - \frac{i}{2} [(q_{n-k}^* y^{(k)})^{(k-1)} + (q_{n-k} y^{(k-1)})^{(k)}]) + p_n y, \quad (2.26)$$

be a differential expression of an even order $2n$ with smooth enough operator-valued coefficients $p_k(\cdot), q_k(\cdot) : \Delta \rightarrow [H]$ such that $p_k(t) = p_k^*(t)$ and $0 \in \rho(p_0(t))$. Denote by $y^{[k]}(\cdot)$, $k = 0 \div 2n$ the quasi-derivatives of a vector-function $y(\cdot) : \Delta \rightarrow H$, corresponding to the expression (2.26) and let $\mathcal{D}(l)$ be the set of functions $y(\cdot)$ for which this expression makes sense [20, 21, 22]. With every function $y \in \mathcal{D}(l)$ we associate the functions $y^{(j)}(\cdot) : \Delta \rightarrow H^n$, $j \in \{1, 2\}$ and $\tilde{y}(\cdot) : \Delta \rightarrow H^n \oplus H^n$ by setting

$$y^{(1)}(t) := \{y^{[k-1]}(t)\}_{k=1}^n (\in H^n), \quad y^{(2)}(t) := \{y^{[2n-k]}(t)\}_{k=1}^n (\in H^n), \quad (2.27)$$

$$\tilde{y}(t) = \{y^{(1)}(t), y^{(2)}(t)\} (\in H^n \oplus H^n), \quad t \in \Delta. \quad (2.28)$$

Let \mathcal{K} be a Hilbert space and let $Y(\cdot) : \Delta \rightarrow [\mathcal{K}, H]$ be an operator solution of the differential equation

$$l[y] - \lambda y = 0. \quad (2.29)$$

With each such a solution we associate the operator-functions $Y^{(j)}(\cdot) : \Delta \rightarrow [\mathcal{K}, H^n]$, $j \in \{1, 2\}$ and $\tilde{Y}(\cdot) : \Delta \rightarrow [\mathcal{K}, H^n \oplus H^n]$,

$$Y^{(1)}(t) = (Y(t) \ Y^{[1]}(t) \ \dots \ Y^{[n-1]}(t))^T, \quad Y^{(2)}(t) = (Y^{[2n-1]}(t) \ Y^{[2n-2]}(t) \ \dots \ Y^{[n]}(t))^T, \\ \tilde{Y}(t) = (Y^{(1)}(t) \ Y^{(2)}(t))^T : \mathcal{K} \rightarrow H^n \oplus H^n, \quad t \in \Delta,$$

where $Y^{[k]}(\cdot)$, $k = 0 \div 2n - 1$ are quasi-derivatives of $Y(\cdot)$.

In what follows $\mathfrak{H}(= L_2(\Delta; H))$ is a Hilbert space of all measurable functions $f(\cdot) : \Delta \rightarrow H$ such that $\int_0^b \|f(t)\|^2 dt < \infty$. Moreover $L'_2[\mathcal{K}, H]$ stands for the set of all operator-functions $Y(\cdot) : \Delta \rightarrow [\mathcal{K}, H]$ such that $Y(t)h \in \mathfrak{H}$ for all $h \in \mathcal{K}$.

It is known [20, 21, 22] that the expression (2.26) generates the maximal operator L in \mathfrak{H} , defined on the domain $\mathcal{D} = \mathcal{D}(L) := \{y \in \mathcal{D}(l) \cap \mathfrak{H} : l[y] \in \mathfrak{H}\}$ by $Ly = l[y]$, $y \in \mathcal{D}$. Moreover the Lagrange's identity

$$(Ly, z)_{\mathfrak{H}} - (y, Lz)_{\mathfrak{H}} = [y, z](b) - [y, z](0), \quad y, z \in \mathcal{D} \quad (2.30)$$

holds with

$$[y, z](t) = (y^{(1)}(t), z^{(2)}(t))_{H^n} - (y^{(2)}(t), z^{(1)}(t))_{H^n}, \quad [y, z](b) = \lim_{t \uparrow b} [y, z](t). \quad (2.31)$$

Let $\mathcal{D}_0 = \{y \in \mathcal{D} : \tilde{y}(0) = 0 \text{ and } [y, z](b) = 0, z \in \mathcal{D}\}$ and let $L_0 = L \upharpoonright \mathcal{D}_0$ be the minimal operator generated by the expression (2.26). Then L_0 is a closed densely defined symmetric operator in \mathfrak{H} and $L_0^* = L$ [20, 21, 22]. Moreover the deficiency indices $n_{\pm}(L_0)$ of the operator L_0 are not necessarily equal.

Let $\theta = \theta^* \in \tilde{\mathcal{C}}(H^n)$ and let L_{θ} be a symmetric extension of L_0 with the domain $\mathcal{D}(L_{\theta}) = \{y \in \mathcal{D} : \tilde{y}(0) \in \theta, [y, z](b) = 0 \ \forall z \in \mathcal{D}\}$. According to [17] deficiency indices $n_{\pm}(L_{\theta})$ of an operator L_{θ} do not depend on $\theta (= \theta^*)$, which enables us to introduce the deficiency indices at the right endpoint b by $n_{b\pm} := n_{\pm}(L_{\theta})$.

2.5. Decomposing boundary triplets. Assume that \mathcal{H}'_1 is a subspace in a Hilbert space \mathcal{H}'_0 , $\mathcal{H}'_2 := \mathcal{H}'_0 \ominus \mathcal{H}'_1$, $\Gamma'_0 : \mathcal{D} \rightarrow \mathcal{H}'_0$ and $\Gamma'_1 : \mathcal{D} \rightarrow \mathcal{H}'_1$ are linear maps and P'_j is the orthoprojector in \mathcal{H}'_0 onto \mathcal{H}'_j , $j \in \{1, 2\}$. Moreover let $\mathcal{H}_0 = H^n \oplus \mathcal{H}'_0$, $\mathcal{H}_1 = H^n \oplus \mathcal{H}'_1$ and let $\Gamma_j : \mathcal{D} \rightarrow \mathcal{H}_j$, $j \in \{0, 1\}$ be linear maps given for all $y \in \mathcal{D}$ by

$$\Gamma_0 y = \{y^{(2)}(0), \Gamma'_0 y\} (\in H^n \oplus \mathcal{H}'_0), \quad \Gamma_1 y = \{-y^{(1)}(0), \Gamma'_1 y\} (\in H^n \oplus \mathcal{H}'_1). \quad (2.32)$$

Definition 2.8. [17] A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where Γ_0 and Γ_1 are linear maps (2.32), is said to be a decomposing D -boundary triplet (briefly a decomposing D -triplet) for L if the map $\Gamma' = (\Gamma'_0 \ \Gamma'_1)^\top : \mathcal{D} \rightarrow \mathcal{H}'_0 \oplus \mathcal{H}'_1$ is surjective and the following identity holds

$$[y, z](b) = (\Gamma'_1 y, \Gamma'_0 z) - (\Gamma'_0 y, \Gamma'_1 z) + i(P'_2 \Gamma'_0 y, P'_2 \Gamma'_0 z), \quad y, z \in \mathcal{D}. \quad (2.33)$$

In the case $\mathcal{H}'_0 = \mathcal{H}'_1 =: \mathcal{H}'$ ($\iff \mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$) a decomposing D -triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a decomposing boundary triplet for L . For such a triplet the identity (2.33) takes the form

$$[y, z](b) = (\Gamma'_1 y, \Gamma'_0 z) - (\Gamma'_0 y, \Gamma'_1 z), \quad y, z \in \mathcal{D}. \quad (2.34)$$

As was shown in [17], Lemma 3.4 a decomposing D -triplet (a decomposing boundary triplet) for L is a D -triplet (a boundary triplet) in the sense of Definition 2.5 and Remark 2.7. Moreover a decomposing D -triplet (boundary triplet) for L exists if and only if $n_{b-} \leq n_{b+}$ (respectively, $n_{b-} = n_{b+}$), in which case

$$\dim \mathcal{H}'_1 = n_{b-} \leq n_{b+} = \dim \mathcal{H}'_0, \quad \dim \mathcal{H}_1 = n_-(L_0) \leq n_+(L_0) = \dim \mathcal{H}_0 \quad (2.35)$$

(respectively, $n_{b-} = n_{b+} = \dim \mathcal{H}'$ and $n_-(L_0) = n_+(L_0) = \dim \mathcal{H}$). Therefore in the sequel we suppose (without loss of generality) that $n_{b-} \leq n_{b+}$ and, consequently, $n_-(L_0) \leq n_+(L_0)$.

Proposition 2.9. [17] Let $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D -triplet (2.32) for L and let $\gamma_\pm(\cdot)$ be the corresponding γ -fields (2.18). Then:

1) For each $\lambda \in \mathbb{C}_+$ ($z \in \mathbb{C}_-$) there exists a unique operator function $Z_+(\cdot, \lambda) \in L'_2[\mathcal{H}_0, H]$ ($Z_-(\cdot, z) \in L'_2[\mathcal{H}_1, H]$), satisfying (2.29) and the boundary condition $\Gamma_0(Z_+(t, \lambda)h_0) = h_0$, $h_0 \in \mathcal{H}_0$ (resp. $P_1\Gamma_0(Z_-(t, z)h_1) = h_1$, $h_1 \in \mathcal{H}_1$). If

$$Z_+(t, \lambda) = (v_0(t, \lambda) \ u_+(t, \lambda)) : H^n \oplus \mathcal{H}'_0 \rightarrow H, \quad \lambda \in \mathbb{C}_+, \quad (2.36)$$

$$Z_-(t, z) = (v_0(t, z) \ u_-(t, z)) : H^n \oplus \mathcal{H}'_1 \rightarrow H, \quad z \in \mathbb{C}_- \quad (2.37)$$

are the block representations of $Z_+(\cdot, \lambda)$ and $Z_-(\cdot, z)$, then the above boundary condition can be represented as

$$v_0^{(2)}(0, \mu) = I_{H^n} \quad (\mu \in \mathbb{C} \setminus \mathbb{R}); \quad \Gamma'_0(v_0(t, \lambda)\hat{h}) = 0, \quad P'_1\Gamma'_0(v_0(t, z)\hat{h}) = 0, \quad \hat{h} \in H^n$$

$$u_+^{(2)}(0, \lambda) = 0; \quad \Gamma'_0(u_+(t, \lambda)h'_0) = h'_0, \quad \lambda \in \mathbb{C}_+, \quad h'_0 \in \mathcal{H}'_0;$$

$$u_-^{(2)}(0, z) = 0, \quad P'_1\Gamma'_0(u_-(t, z)h'_1) = h'_1, \quad z \in \mathbb{C}_-, \quad h'_1 \in \mathcal{H}'_1.$$

2) for all $\lambda \in \mathbb{C}_+$ and $z \in \mathbb{C}_-$ the following equalities hold

$$(\gamma_+(\lambda)h_0)(t) = Z_+(t, \lambda)h_0, \quad h_0 \in \mathcal{H}_0 \quad (\gamma_-(z)h_1)(t) = Z_-(t, z)h_1, \quad h_1 \in \mathcal{H}_1. \quad (2.38)$$

Next assume that $M_{\pm}(\cdot)$ are the Weyl functions (2.19), (2.20) corresponding to the D -triplet Π and let

$$M_+(\lambda) = \begin{pmatrix} m(\lambda) & M_{2+}(\lambda) \\ M_{3+}(\lambda) & M_{4+}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_0 \rightarrow H^n \oplus \mathcal{H}'_1, \quad \lambda \in \mathbb{C}_+ \quad (2.39)$$

$$M_-(z) = \begin{pmatrix} m(z) & M_{2-}(z) \\ M_{3-}(z) & M_{4-}(z) \end{pmatrix} : H^n \oplus \mathcal{H}'_1 \rightarrow H^n \oplus \mathcal{H}'_0, \quad z \in \mathbb{C}_- \quad (2.40)$$

be their block representations. As was proved in [17, Theorem 3.12] all the entries in (2.19) and (2.20) can be defined immediately in terms of boundary values of the functions $v_0(\cdot, \lambda)$ and $u_{\pm}(\cdot, \lambda)$. In particular formulas (2.39) and (2.40) generate the uniformly strict Nevanlinna function $m(\lambda) = -v_0^{(1)}(0, \lambda)$, which we called in [17] the m -function.

2.6. Generalized resolvents and characteristic matrices. Let $\tilde{A} \supset L_0$ be an exit space selfadjoint extension of the operator L_0 acting in the Hilbert space $\tilde{\mathfrak{H}} \supset \mathfrak{H}$ and let $\tilde{E}(t)$ be the orthogonal spectral function of the operator \tilde{A} . Recall that the operator functions $R(\lambda) = P_{\mathfrak{H}}(\tilde{A} - \lambda)^{-1}|_{\mathfrak{H}}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $F(t) = P_{\mathfrak{H}}\tilde{E}(t)|_{\mathfrak{H}}$ are called generalized resolvent and spectral function of the operator L_0 respectively. In the sequel we suppose that the spectral function $\tilde{E}(t)$ (or equivalently the extension \tilde{A}) is minimal, which means that $\text{span}\{\mathfrak{H}, \tilde{E}(t)\mathfrak{H} : t \in \mathbb{R}\} = \tilde{\mathfrak{H}}$.

Let $Y_0(\cdot, \lambda) : \Delta \rightarrow [H^n \oplus H^n, H]$ be the "canonical" operator solution of the equation (2.29) with the initial data $\tilde{Y}_0(0, \lambda) = I_{H^n \oplus H^n}$ and let

$$J_{H^n} := \begin{pmatrix} 0 & -I_{H^n} \\ I_{H^n} & 0 \end{pmatrix} : H^n \oplus H^n \rightarrow H^n \oplus H^n. \quad (2.41)$$

According to [24, 3] the generalized resolvent $R(\lambda)$ admits the representation

$$(R(\lambda)f)(x) = \int_0^b G(x, t, \lambda)f(t) dt := \lim_{\eta \uparrow b} \int_0^{\eta} G(x, t, \lambda)f(t) dt, \quad f = f(\cdot) \in \mathfrak{H} \quad (2.42)$$

with the Green function $G(\cdot, \cdot, \lambda) : \Delta \times \Delta \rightarrow [H]$ given by

$$G(x, t, \lambda) = Y_0(x, \lambda)(\Omega(\lambda) + \frac{1}{2} \text{sgn}(t - x)J_{H^n})Y_0^*(t, \bar{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (2.43)$$

Here $\Omega(\lambda) \in [H^n \oplus H^n]$ is a Nevanlinna operator function, which is called a characteristic matrix of the generalized resolvent $R(\lambda)$ [24].

Next assume that $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a decomposing D -triplet (2.32) for L and $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ is a collection of holomorphic pairs (2.13) with the block representations

$$C_0(\lambda) = (\hat{C}_0(\lambda) \ C'_0(\lambda)) : H^n \oplus \mathcal{H}'_0 \rightarrow \mathcal{K}_0, \quad (2.44)$$

$$C_1(\lambda) = (\hat{C}_1(\lambda) \ C'_1(\lambda)) : H^n \oplus \mathcal{H}'_1 \rightarrow \mathcal{K}_0, \quad \lambda \in \mathbb{C}_+$$

$$D_0(\lambda) = (\hat{D}_0(\lambda) \ D'_0(\lambda)) : H^n \oplus \mathcal{H}'_0 \rightarrow \mathcal{K}_1, \quad (2.45)$$

$$D_1(\lambda) = (\hat{D}_1(\lambda) \ D'_1(\lambda)) : H^n \oplus \mathcal{H}'_1 \rightarrow \mathcal{K}_1, \quad \lambda \in \mathbb{C}_-$$

For a given function $f \in \mathfrak{H}$ consider the boundary value problem

$$l[y] - \lambda y = f \quad (2.46)$$

$$\hat{C}_0(\lambda)y^{(2)}(0) + \hat{C}_1(\lambda)y^{(1)}(0) + C'_0(\lambda)\Gamma'_0 y - C'_1(\lambda)\Gamma'_1 y = 0, \quad \lambda \in \mathbb{C}_+ \quad (2.47)$$

$$\hat{D}_0(\lambda)y^{(2)}(0) + \hat{D}_1(\lambda)y^{(1)}(0) + D'_0(\lambda)\Gamma'_0 y - D'_1(\lambda)\Gamma'_1 y = 0, \quad \lambda \in \mathbb{C}_-. \quad (2.48)$$

In view of (2.44) and (2.45) the conditions (2.47) and (2.48) can be written as

$$C_0(\lambda)\Gamma_0 y - C_1(\lambda)\Gamma_1 y = 0, \quad \lambda \in \mathbb{C}_+; \quad D_0(\lambda)\Gamma_0 y - D_1(\lambda)\Gamma_1 y = 0, \quad \lambda \in \mathbb{C}_-. \quad (2.49)$$

A function $y(\cdot, \cdot) : \Delta \times (\mathbb{C} \setminus \mathbb{R}) \rightarrow H$ is called a solution of the boundary problem (2.46)–(2.48) if for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the function $y(\cdot, \lambda)$ belongs to \mathcal{D} and satisfies the equation (2.46) and the boundary conditions (2.47), (2.48).

Theorem 2.10. [19] *Let $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ be a collection given by (2.13) and (2.44), (2.45) and let $\tilde{\Omega}_{\tau+}(\lambda)$ and $\tilde{\Omega}_{\tau-}(\lambda)$ be the operator function defined by*

$$\tilde{\Omega}_{\tau+}(\lambda) = \begin{pmatrix} \tilde{\omega}_{1+}(\lambda) & \tilde{\omega}_{2+}(\lambda) \\ \tilde{\omega}_{3+}(\lambda) & \tilde{\omega}_{4+}(\lambda) \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_0, \quad \lambda \in \mathbb{C}_+, \quad (2.50)$$

$$\tilde{\omega}_{1+}(\lambda) = M_+(\lambda) - M_+(\lambda)(\tau_+(\lambda) + M_+(\lambda))^{-1}M_+(\lambda) \quad (2.51)$$

$$\tilde{\omega}_{2+}(\lambda) = -\frac{1}{2}I_{\mathcal{H}_1} + M_+(\lambda)(\tau_+(\lambda) + M_+(\lambda))^{-1} \quad (2.52)$$

$$\tilde{\omega}_{3+}(\lambda) = -\frac{1}{2}I_{\mathcal{H}_0} + (\tau_+(\lambda) + M_+(\lambda))^{-1}M_+(\lambda) \quad (2.53)$$

$$\tilde{\omega}_{4+}(\lambda) = -(\tau_+(\lambda) + M_+(\lambda))^{-1}, \quad (2.54)$$

$$\tilde{\Omega}_{\tau-}(\lambda) = (\tilde{\Omega}_{\tau+}(\bar{\lambda}))^*, \quad \lambda \in \mathbb{C}_-. \quad (2.55)$$

Then:

1) for each $f \in \mathfrak{H}$ the boundary problem (2.46)–(2.48) has the unique solution $y(t, \lambda) = y_f(t, \lambda)$ and the equality $(R(\lambda)f)(t) = y_f(t, \lambda)$, $f \in \mathfrak{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ defines a generalized resolvent $R(\lambda) := R_\tau(\lambda)$ of the minimal operator L_0 ;

2) the characteristic matrix of the generalized resolvent $R_\tau(\lambda)$ is

$$\Omega(\lambda) = \Omega_\tau(\lambda) := P_{H^n \oplus H^n} \tilde{\Omega}_{\tau+}(\lambda) \upharpoonright H^n \oplus H^n, \quad \lambda \in \mathbb{C}_+. \quad (2.56)$$

Conversely for each generalized resolvent $R(\lambda)$ there exists the unique $\tau \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ such that $R(\lambda) = R_\tau(\lambda)$. Moreover $R_\tau(\lambda)$ is a canonical resolvent if and only if $\tau \in \tilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$.

Proposition 2.11. *Assume that $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ is a collection given by (2.44) and (2.45), $\tilde{D}_1(\lambda) \in [\mathcal{H}_1, \mathcal{K}_-]$ and $\tilde{D}_0(\lambda) \in [\mathcal{H}_0, \mathcal{K}_-]$ are defined by*

$$\tilde{D}_1(\lambda) := D_0(\lambda) \upharpoonright \mathcal{H}_1, \quad \tilde{D}_0(\lambda) = D_1(\lambda)P_1 + iD_0(\lambda)P_2, \quad \lambda \in \mathbb{C}_-, \quad (2.57)$$

and $D_{01}(\lambda)$, $D_{02}(\lambda)$ are entries of the block representation (2.6). Moreover let $\gamma_\tau(\lambda) \in [H^n \oplus H^n, \mathfrak{H}]$ and $\tilde{\alpha}(\lambda) \in [\mathcal{K}_-, H^n \oplus H^n]$ be operator functions given by

the block matrix representations

$$\gamma_\tau(\lambda) = \gamma_+(\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}(-\hat{C}_0(\lambda) : \hat{C}_1(\lambda)), \quad \lambda \in \mathbb{C}_+ \quad (2.58)$$

$$\tilde{\alpha}(\lambda) = \begin{pmatrix} -P_{H^n}M_-(\lambda) \\ P_{H^n} \end{pmatrix} (\tilde{D}_1(\lambda) - \tilde{D}_0(\lambda)M_-(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_-. \quad (2.59)$$

Then the corresponding characteristic matrix $\Omega_\tau(\cdot)$ satisfies the identity

$$\begin{aligned} \Omega_\tau(\mu) - \Omega_\tau^*(\lambda) &= (\mu - \bar{\lambda})\gamma_\tau^*(\lambda)\gamma_\tau(\mu) - \tilde{\alpha}(\bar{\lambda})(D_1(\bar{\lambda})D_{01}^*(\bar{\mu}) - D_{01}(\bar{\lambda})D_1^*(\bar{\mu}) + \\ &\quad + iD_{02}(\bar{\lambda})D_{02}^*(\bar{\mu}))\tilde{\alpha}^*(\bar{\mu}), \quad \mu, \lambda \in \mathbb{C}_+. \end{aligned} \quad (2.60)$$

Moreover the following equality holds

$$s - \lim_{y \rightarrow \infty} \Omega_\tau(iy)/y = 0 \quad (2.61)$$

Proof. The identity (2.60) was proved in [19]. To prove (2.61) assume that

$$\Omega_\tau(\lambda) = \begin{pmatrix} \omega_1(\lambda) & \omega_2(\lambda) \\ \omega_3(\lambda) & \omega_4(\lambda) \end{pmatrix} : H^n \oplus H^n \rightarrow H^n \oplus H^n, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \quad (2.62)$$

is the block matrix representation of $\Omega_\tau(\lambda)$. Then by (2.51) and (2.54)

$$\begin{aligned} \omega_1(\lambda) &= P_{H^n}(\tilde{\omega}_{1+}(\lambda) \upharpoonright \mathcal{H}_1) \upharpoonright H^n = \\ &= P_{H^n}(M(\lambda) - M_+(\lambda)(\tau_+(\lambda) + M_+(\lambda))^{-1}M(\lambda)) \upharpoonright H^n, \\ \omega_4(\lambda) &= P_{H^n}(P_1 \tilde{\omega}_{4+}(\lambda)) \upharpoonright H^n = -P_{H^n}(P_1(\tau_+(\lambda) + M_+(\lambda))^{-1}) \upharpoonright H^n, \quad \lambda \in \mathbb{C}_+, \end{aligned}$$

which in view of (2.23) and (2.24) gives

$$s - \lim_{y \rightarrow \infty} \omega_1(iy)/y = s - \lim_{y \rightarrow \infty} \omega_4(iy)/y = 0.$$

This and the representation (2.62) proves the equality (2.61). \square

Remark 2.12. It follows from Theorem 2.10 that the boundary problem (2.46)-(2.48) gives a parameterization of all generalized resolvents $R(\lambda) = R_\tau(\lambda)$ and characteristic matrices $\Omega(\lambda) = \Omega_\tau(\lambda)$ by means of the Nevanlinna boundary parameter τ . Moreover since a spectral function $F(t)$ is uniquely defined by the corresponding generalized resolvent $R(\lambda)$, one obtains the parameterization $F(t) = F_\tau(t)$ of all spectral functions of the operator L_0 by means of the same boundary parameter τ .

3. m -FUNCTIONS AND CHARACTERISTIC MATRICES

3.1. Quasi-constant and N -triangular Nevanlinna collections. Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D -triplet (2.32) for L (with $\mathcal{H}_j = H^n \oplus \mathcal{H}'_j$, $j \in \{0, 1\}$). A Nevanlinna collection $\{C(\cdot), D(\cdot)\}$ defined by (2.3), (2.4) and the block representations (2.44), (2.45) will be called quasi-constant if $\hat{C}_j(\lambda) = \hat{D}_j(z) \equiv \hat{C}_j(\in) \in [H^n, \mathcal{K}_1]$, $j \in \{0, 1\}$ for all $\lambda \in \mathbb{C}_+$ and $z \in \mathbb{C}_-$ (such a definition is correct, since $\mathcal{K}_1 \subset \mathcal{K}_0$). Clearly, each constant pair $\theta(= \theta^*) = \{(C_0, C_1); \mathcal{K}\}$ is quasi-constant.

Next assume that

$$N = (N_0 \ N_1) : H^n \oplus H^n \rightarrow \hat{\mathcal{K}} \quad (3.1)$$

is an admissible operator pair (that is $\mathcal{R}(N) = \hat{\mathcal{K}}$) and let $\theta_N \in \tilde{\mathcal{C}}(H^n)$ be a linear relation given by $\theta_N = \{(N_0, N_1); \hat{\mathcal{K}}\}$. The operator pair (3.1) will be called symmetric (selfadjoint) if the linear relation θ_N is symmetric (selfadjoint).

Definition 3.1. A Nevanlinna collection $\{C(\cdot), D(\cdot)\}$ defined by (2.3), (2.4) will be called N -triangular if there exist a Hilbert space \mathcal{K}'_0 and a subspace $\mathcal{K}'_1 \subset \mathcal{K}'_0$ such that $\mathcal{K}_j = \hat{\mathcal{K}} \oplus \mathcal{K}'_j$, $j \in \{0, 1\}$ and the following block representations hold

$$C_0(\lambda) = \begin{pmatrix} N_0 & C'_{01}(\lambda) \\ 0 & C'_{02}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_0 \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}'_0, \quad \lambda \in \mathbb{C}_+ \quad (3.2)$$

$$C_1(\lambda) = \begin{pmatrix} N_1 & C'_{11}(\lambda) \\ 0 & C'_{12}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_1 \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}'_0, \quad \lambda \in \mathbb{C}_+ \quad (3.3)$$

$$D_0(\lambda) = \begin{pmatrix} N_0 & D'_{01}(\lambda) \\ 0 & D'_{02}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_0 \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}'_1, \quad \lambda \in \mathbb{C}_- \quad (3.4)$$

$$D_1(\lambda) = \begin{pmatrix} N_1 & D'_{11}(\lambda) \\ 0 & D'_{12}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_1 \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}'_1, \quad \lambda \in \mathbb{C}_- \quad (3.5)$$

A constant N -triangular collection can be regarded as an operator pair

$$C = (C_0 \ C_1) : (H^n \oplus \mathcal{H}'_0) \oplus (H^n \oplus \mathcal{H}'_1) \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}' \quad (3.6)$$

defined by the block matrix representations

$$C_0 = \begin{pmatrix} N_0 & C'_{01} \\ 0 & C'_{02} \end{pmatrix} : H^n \oplus \mathcal{H}'_0 \rightarrow \underbrace{\hat{\mathcal{K}} \oplus \mathcal{K}'}_{\mathcal{K}}, \quad C_1 = \begin{pmatrix} N_1 & C'_{11} \\ 0 & C'_{12} \end{pmatrix} : H^n \oplus \mathcal{H}'_1 \rightarrow \underbrace{\hat{\mathcal{K}} \oplus \mathcal{K}'}_{\mathcal{K}} \quad (3.7)$$

and satisfying the relations (2.11).

Assume now that $\{C(\cdot), D(\cdot)\}$ is a quasi-constant Nevanlinna collection defined by (2.3), (2.4) and (2.44), (2.45) and let $\hat{\mathcal{K}}(\subset \mathcal{K}_1)$ be the range of the operator

$$\hat{C} := C(\lambda) \upharpoonright H^n \oplus H^n = (\hat{C}_0 \ \hat{C}_1) : H^n \oplus H^n \rightarrow \mathcal{K}_1. \quad (3.8)$$

It is clear that the collection $\{C(\cdot), D(\cdot)\}$ is N -triangular with some N if and only if $\hat{\mathcal{K}}$ is closed, in which case $N_j = \hat{C}_j(\in [H^n, \hat{\mathcal{K}}])$, $j \in \{0, 1\}$ (here \hat{C}_j is considered as acting from H^n to $\hat{\mathcal{K}}$). In this connection the following proposition holds.

Proposition 3.2. *If $n_{b+} < \infty$ (in particular, $\dim H < \infty$), then each quasi-constant Nevanlinna collection is N -triangular.*

Proof. Since the operator pair (2.3) is admissible, it follows that $\mathcal{R}(C(\lambda)) = \mathcal{K}_0$ and, therefore, the range of the operator $C^*(\lambda)$ is a closed subspace in $\mathcal{H}_0 \oplus \mathcal{H}_1$. Moreover by (3.8) $\hat{C}^* = P_{H^n \oplus H^n} C^*(\lambda)$ and, consequently,

$$\mathcal{R}(\hat{C}^*) = P_{H^n \oplus H^n} \mathcal{R}(C^*(\lambda)). \quad (3.9)$$

Since in view of (2.35) $\text{codim}(H^n \oplus H^n) = \dim(\mathcal{H}'_0 \oplus \mathcal{H}'_1) < \infty$, it follows from (3.9) that $\mathcal{R}(\hat{C}^*)$ is a closed subspace in $H^n \oplus H^n$. This implies that $\hat{\mathcal{K}}(=\mathcal{R}(\hat{C}))$ is also closed. \square

Remark 3.3. In the case $n_{b+} = \infty (\Leftrightarrow \dim \mathcal{H}'_0 = \infty)$ one can easily construct a quasi-constant (and even constant) Nevanlinna collection $\{C(\cdot), D(\cdot)\}$ with not closed subspace $\hat{\mathcal{K}}$, which implies that this collection is not N -triangular with any N . Hence the condition $n_{b+} < \infty$ in Proposition 3.2 is essential.

Two N -triangular Nevanlinna collections $\{C(\cdot), D(\cdot)\}$ and $\{\tilde{C}(\cdot), \tilde{D}(\cdot)\}$ (with the same N) are said to be equivalent if the operator pairs $C(\cdot)$ and $\tilde{C}(\cdot)$ as well as $D(\cdot)$ and $\tilde{D}(\cdot)$ are equivalent in the sense of Definition 2.1. It is clear that for a given operator pair N (see (3.1)) the set of all N -triangular Nevanlinna collections falls into nonintersecting equivalence classes. In what follows the set of all such classes will be denoted by $TR\{\mathcal{H}_0, \mathcal{H}_1\}$. Moreover we will denote by $\mathcal{P} = \{C(\cdot), D(\cdot)\}$ both an N -triangular Nevanlinna collection and the corresponding equivalence class.

Definition 3.4. A collection (the corresponding equivalence class) $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ is said to belong to the class $TR^0\{\mathcal{H}_0, \mathcal{H}_1\}$ if it admits the representation (3.6), (3.7) as a constant N -triangular collection.

In the sequel we write $\mathcal{P} = \{C_0, C_1\} \in TR^0\{\mathcal{H}_0, \mathcal{H}_1\}$ identifying the collection $\mathcal{P} \in TR^0\{\mathcal{H}_0, \mathcal{H}_1\}$ and the corresponding operator pair (3.6), (3.7).

In the case $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ (i.e., in the case of a decomposing boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$) we let $TR(\mathcal{H}) := TR(\mathcal{H}, \mathcal{H})$ and $TR^0(\mathcal{H}) := TR^0(\mathcal{H}, \mathcal{H})$.

Proposition 3.5. Assume that $N = (N_0 \ N_1)$ is an operator pair (3.1) and $\{C(\cdot), D(\cdot)\} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ is a collection (3.2)-(3.5). Then the pair N is symmetric and

$$n \dim H \leq \dim \hat{\mathcal{K}} \leq n_-(L_0); \quad (3.10)$$

Proof. Let $\tau_{\pm}(\lambda)$ be linear relations (2.13). Then in view of (3.2)-(3.5) $\theta_N = \tau_{\pm}(\lambda) \cap (H^n \oplus H^n)$, $\lambda \in \mathbb{C}_{\pm}$ and (2.14) shows that θ_N is a symmetric linear relation. Therefore $\dim H^n \leq \text{codim } \theta_N = \dim \hat{\mathcal{K}}$, which together with (2.15) and the second relation in (2.35) gives (3.10). \square

3.2. Generalized resolvents and the Green function. Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D -triplet (2.32) for L and let $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ be a collection (3.2)-(3.5). Then the corresponding boundary problem (2.46)-(2.48) can be written as

$$l[y] - \lambda y = f \quad (3.11)$$

$$N_0 y^{(2)}(0) + N_1 y^{(1)}(0) + C'_{01}(\lambda) \Gamma'_0 y - C'_{11}(\lambda) \Gamma'_1 y = 0, \quad (3.12)$$

$$C'_{02}(\lambda) \Gamma'_0 y - C'_{12}(\lambda) \Gamma'_1 y = 0, \quad \lambda \in \mathbb{C}_+; \quad (3.13)$$

$$N_0 y^{(2)}(0) + N_1 y^{(1)}(0) + D'_{01}(\lambda) \Gamma'_0 y - D'_{11}(\lambda) \Gamma'_1 y = 0, \quad (3.14)$$

$$D'_{02}(\lambda) \Gamma'_0 y - D'_{12}(\lambda) \Gamma'_1 y = 0, \quad \lambda \in \mathbb{C}_-. \quad (3.15)$$

Moreover in the case $\mathcal{P} = \{C_0, C_1\} \in TR^0\{\mathcal{H}_0, \mathcal{H}_1\}$ (see (3.6) and (3.7)) the boundary conditions (3.12)-(3.15) take the form

$$N_0 y^{(2)}(0) + N_1 y^{(1)}(0) + C'_{01} \Gamma'_0 y - C'_{11} \Gamma'_1 y = 0 \quad (3.16)$$

$$C'_{02} \Gamma'_0 y - C'_{12} \Gamma'_1 y = 0 \quad (3.17)$$

The following corollary is immediate from Theorem 2.10.

Corollary 3.6. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D -triplet (2.32) for L and let $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ be a collection given by (3.2)-(3.5). Then the boundary problem (3.11)-(3.15) generates the generalized resolvent $R(\lambda) = R_{\mathcal{P}}(\lambda)$ of the operator L_0 (in the same way as in Theorem 2.10). Moreover $R(\lambda)$ is a canonical resolvent if and only if $\mathcal{P} \in TR^0\{\mathcal{H}_0, \mathcal{H}_1\}$, in which case the corresponding boundary conditions can be defined by (3.16) and (3.17).*

Remark 3.7. Note that in view of Corollary 3.6 the generalized resolvent $R(\lambda) = R_{\mathcal{P}}(\lambda)$ can be also defined by $R_{\mathcal{P}}(\lambda) = (\tilde{A}(\lambda) - \lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ where $\tilde{A}(\lambda) = L \upharpoonright \mathcal{D}(\tilde{A}(\lambda))$ and $\mathcal{D}(\tilde{A}(\lambda))$ is the set of all functions $y \in \mathcal{D}$ satisfying the boundary conditions (3.12)-(3.15) or, equivalently, (2.49).

Assume that $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ is a collection (3.2)-(3.5) and let $\tilde{D}_1(\lambda) (\in [\mathcal{H}_1, \mathcal{K}_1])$ and $\tilde{D}_0(\lambda) (\in [\mathcal{H}_0, \mathcal{K}_1])$ be defined by

$$\tilde{D}_1(\lambda) := D_0(\lambda) \upharpoonright \mathcal{H}_1, \quad \tilde{D}_0(\lambda) = D_1(\lambda)P_1 + iD_0(\lambda)P_2, \quad \lambda \in \mathbb{C}_-.$$

It follows from (3.4) and (3.5) that the following block representations hold

$$\begin{aligned} \tilde{D}_1(\lambda) &= \begin{pmatrix} N_0 & \tilde{D}'_{01}(\lambda) \\ 0 & \tilde{D}'_{02}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_1 \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}'_1, \quad \lambda \in \mathbb{C}_- \\ \tilde{D}_0(\lambda) &= \begin{pmatrix} N_1 & \tilde{D}'_{11}(\lambda) \\ 0 & \tilde{D}'_{12}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_0 \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}'_1, \quad \lambda \in \mathbb{C}_-. \end{aligned}$$

Proposition 3.8. *Let the conditions of Corollary 3.6 be satisfied. Then:*

1) *for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists the unique operator function $v(\cdot, \lambda) \in L'_2[\hat{\mathcal{K}}, H]$ satisfying the equation $l[y] - \lambda y = 0$ and the boundary conditions*

$$(N_0 v^{(2)}(0, \lambda) + N_1 v^{(1)}(0, \lambda))\hat{h} + (C'_{01}(\lambda)\Gamma'_0 - C'_{11}(\lambda)\Gamma'_1)(v(t, \lambda)\hat{h}) = \hat{h} \quad (3.18)$$

$$(C'_{02}(\lambda)\Gamma'_0 - C'_{12}(\lambda)\Gamma'_1)(v(t, \lambda)\hat{h}) = 0, \quad \hat{h} \in \hat{\mathcal{K}}, \quad \lambda \in \mathbb{C}_+; \quad (3.19)$$

$$(N_0 v^{(2)}(0, \lambda) + N_1 v^{(1)}(0, \lambda))\hat{h} + (D'_{01}(\lambda)\Gamma'_0 - D'_{11}(\lambda)\Gamma'_1)(v(t, \lambda)\hat{h}) = \hat{h} \quad (3.20)$$

$$(D'_{02}(\lambda)\Gamma'_0 - D'_{12}(\lambda)\Gamma'_1)(v(t, \lambda)\hat{h}) = 0, \quad \hat{h} \in \hat{\mathcal{K}}, \quad \lambda \in \mathbb{C}_-. \quad (3.21)$$

2) *The functions $v(\cdot, \lambda)$ and $Z_{\pm}(\cdot, \lambda)$ (see (2.36) and (2.37)) are connected by*

$$v(t, \lambda) = \begin{cases} Z_+(t, \lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1} \upharpoonright \hat{\mathcal{K}}, & \lambda \in \mathbb{C}_+ \\ Z_-(t, \lambda)(\tilde{D}_1(\lambda) - \tilde{D}_0(\lambda)M_-(\lambda))^{-1} \upharpoonright \hat{\mathcal{K}}, & \lambda \in \mathbb{C}_-, \end{cases} \quad (3.22)$$

where $M_{\pm}(\cdot)$ are the Weyl functions (2.39) and (2.40) for Π .

3) for each $\lambda \in \mathbb{C}_+$ (resp. $\lambda \in \mathbb{C}_-$) the equality $y(t) = v(t, \lambda)\hat{h}$ gives a bijective correspondence between all $\hat{h} \in \hat{\mathcal{K}}$ and all solutions $y(\cdot)$ of the equation (2.29), which belong to \mathfrak{H} and satisfy the boundary condition (3.13) (resp. (3.15)). Therefore the operator function $v_{\mathcal{P}}(\cdot, \lambda)$ is a fundamental solution of the boundary problems (2.29), (3.13) for $\lambda \in \mathbb{C}_+$ and (2.29), (3.15) for $\lambda \in \mathbb{C}_-$ (see [18, 22]).

Proof. 1)-2) It follows from (2.32) and (3.2)-(3.5) that the conditions (3.18)-(3.21) are equivalent to

$$(C_0(\lambda)\Gamma_0 - C_1(\lambda)\Gamma_1)(v(t, \lambda)\hat{h}) = \hat{h}, \quad \hat{h} \in \hat{\mathcal{K}}, \quad \lambda \in \mathbb{C}_+ \quad (3.23)$$

$$(D_0(\lambda)\Gamma_0 - D_1(\lambda)\Gamma_1)(v(t, \lambda)\hat{h}) = \hat{h}, \quad \hat{h} \in \hat{\mathcal{K}}, \quad \lambda \in \mathbb{C}_-. \quad (3.24)$$

As was shown in [19] $0 \in \rho(C_0(\lambda) - C_1(\lambda)M_+(\lambda))$, $0 \in \rho(\tilde{D}_1(\lambda) - \tilde{D}_0(\lambda)M_-(\lambda))$ and

$$(C_0(\lambda)\Gamma_0 - C_1(\lambda)\Gamma_1)(Z_+(t, \lambda)h) = (C_0(\lambda) - C_1(\lambda)M_+(\lambda))h, \quad h \in \mathcal{H}_0, \quad \lambda \in \mathbb{C}_+ \quad (3.25)$$

$$(D_0(\lambda)\Gamma_0 - D_1(\lambda)\Gamma_1)(Z_-(t, \lambda)h) = (\tilde{D}_1(\lambda) - \tilde{D}_0(\lambda)M_-(\lambda))h, \quad h \in \mathcal{H}_1, \quad \lambda \in \mathbb{C}_-. \quad (3.26)$$

Hence the equality (3.22) correctly defines the function $v(\cdot, \lambda) \in L'_2[\hat{\mathcal{K}}, H]$ satisfying (3.23), (3.24) and consequently (3.18)-(3.21). The uniqueness of such a function follows from the inclusion $\lambda \in \rho(\tilde{A}(\lambda))$, where $\tilde{A}(\lambda)$ is defined in Remark 3.7.

3) If $\hat{h} \in \hat{\mathcal{K}}$, then by the statement 1) the function $y(t) = v(t, \lambda)\hat{h}$ satisfies the equation (2.29) and the conditions (3.13), (3.15). Conversely let $\lambda \in \mathbb{C}_+$ and a function $y \in \mathcal{D}$ satisfies (2.29) and (3.13). Then there exists $h = \{\hat{h}, h'\} \in \hat{\mathcal{K}} \oplus \mathcal{K}'$ such that $y = Z_+(t, \lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}h$ and by (3.25) one has $(C_0(\lambda)\Gamma_0 - C_1(\lambda)\Gamma_1)y = h$. Therefore in view of (3.13) $h' = 0$, so that $h = \hat{h} \in \hat{\mathcal{K}}$ and by (3.22) $y = v(t, \lambda)\hat{h}$. Similarly by using (3.26) one proves the same statement in the case $\lambda \in \mathbb{C}_-$. \square

Remark 3.9. One can easily verify that for a given operator pair $N = (N_0 \ N_1)$ the operator function $v(\cdot, \lambda)$ is uniquely defined by the equivalence class $\mathcal{P} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$, i.e., $v(\cdot, \lambda)$ does not depend on the choice of an N -triangular Nevanlinna collection (3.2)-(3.5) inside the equivalence class. To emphasize this fact we will write $v(\cdot, \lambda) = v_{\mathcal{P}}(\cdot, \lambda)$.

Theorem 3.10. Assume that $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a decomposing D -triplet (2.32) for L , $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ is a collection (3.2)-(3.5) and $\varphi_N(\cdot, \lambda) : \Delta \rightarrow [\hat{\mathcal{K}}, H]$, $\lambda \in \mathbb{C}$ is the operator solution of (2.29) with the initial data

$$\varphi_N^{(1)}(0, \lambda) = -N_0^*, \quad \varphi_N^{(2)}(0, \lambda) = N_1^*, \quad \lambda \in \mathbb{C}. \quad (3.27)$$

Then the generalized resolvent $R(\lambda) = R_{\mathcal{P}}(\lambda)$ generated by the boundary problem (3.11)-(3.15) admits the representation (2.42) with the Green function $G(x, t, \lambda) =$

$G_{\mathcal{P}}(x, t, \lambda)$ given by

$$G_{\mathcal{P}}(x, t, \lambda) = \begin{cases} v_{\mathcal{P}}(x, \lambda) \varphi_N^*(t, \bar{\lambda}), & x > t \\ \varphi_N(x, \lambda) v_{\mathcal{P}}^*(t, \bar{\lambda}), & x < t \end{cases}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.28)$$

Proof. Let $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ be a collection given by (2.13) and (2.44), (2.45), and let $Y_+(\cdot, \lambda) : \Delta \rightarrow [\mathcal{K}_1, H]$, $\lambda \in \mathbb{C}_+$, and $Y_-(\cdot, z) : \Delta \rightarrow [\mathcal{K}_0, H]$, $z \in \mathbb{C}_-$, be the operator solutions of the equation (2.29) with the initial data

$$\tilde{Y}_+(0, \lambda) = (-\hat{D}_0^*(\bar{\lambda}) \ \hat{D}_1^*(\bar{\lambda}))^\top, \quad \tilde{Y}_-(0, z) = (-\hat{C}_0^*(\bar{z}) \ \hat{C}_1^*(\bar{z}))^\top. \quad (3.29)$$

Assume also that $\mathcal{Z}_+(\cdot, \lambda) \in L'_2[\mathcal{K}_0, H]$ and $\mathcal{Z}_-(\cdot, z) \in L'_2[\mathcal{K}_1, H]$ are given by

$$\mathcal{Z}_+(t, \lambda) = Z_+(t, \lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_+ \quad (3.30)$$

$$\mathcal{Z}_-(t, z) = Z_-(t, z)(\tilde{D}_1(z) - \tilde{D}_0(z)M_-(z))^{-1}, \quad z \in \mathbb{C}_- \quad (3.31)$$

and let

$$Y(t, \lambda) = \begin{cases} Y_+(t, \lambda), & \lambda \in \mathbb{C}_+ \\ Y_-(t, \lambda), & \lambda \in \mathbb{C}_- \end{cases}; \quad \mathcal{Z}(t, \lambda) = \begin{cases} \mathcal{Z}_+(t, \lambda), & \lambda \in \mathbb{C}_+ \\ \mathcal{Z}_-(t, \lambda), & \lambda \in \mathbb{C}_- \end{cases}.$$

Then according to Theorem 14 in [19] the Green function in (2.42) is

$$G(x, t, \lambda) = \begin{cases} \mathcal{Z}(x, \lambda) Y^*(t, \bar{\lambda}), & x > t \\ Y(x, \lambda) \mathcal{Z}^*(t, \bar{\lambda}), & x < t \end{cases}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.32)$$

Next, in the case of the block representations (3.2)-(3.5) one has

$$\hat{C}_j(\lambda) = (N_j \ 0)^\top \in [H^n, \hat{\mathcal{K}} \oplus \mathcal{K}'_0], \quad \hat{D}_j(\lambda) = (N_j \ 0)^\top \in [H^n, \hat{\mathcal{K}} \oplus \mathcal{K}'_1], \quad j \in \{0, 1\}.$$

Therefore the initial data (3.29) can be written in the form $\tilde{Y}_+(0, \lambda) = \begin{pmatrix} -N_0^* & 0 \\ N_1^* & 0 \end{pmatrix} \in [\hat{\mathcal{K}} \oplus \mathcal{K}'_1, H^n \oplus H^n]$, $\tilde{Y}_-(0, z) = \begin{pmatrix} -N_0^* & 0 \\ N_1^* & 0 \end{pmatrix} \in [\hat{\mathcal{K}} \oplus \mathcal{K}'_0, H^n \oplus H^n]$, which in view of (3.27) gives the block representations

$$Y_+(t, \lambda) = (\varphi_N(t, \lambda) \ 0) : \hat{\mathcal{K}} \oplus \mathcal{K}'_1 \rightarrow H, \quad Y_-(t, z) = (\varphi_N(t, z) \ 0) : \hat{\mathcal{K}} \oplus \mathcal{K}'_0 \rightarrow H. \quad (3.33)$$

Moreover by (3.22) the operator functions (3.30) have the block representations

$$\mathcal{Z}_+(t, \lambda) = (v_{\mathcal{P}}(t, \lambda) \ u_+(t, \lambda)), \quad \mathcal{Z}_-(t, z) = (v_{\mathcal{P}}(t, z) \ u_-(t, z)) \quad (3.34)$$

with some operator functions $u_+(t, \lambda)$ and $u_-(t, z)$. Now combining (3.33) and (3.34) with (3.32) we arrive at the equality (3.28). \square

3.3. m -functions. Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D -triplet (2.32) for L and let $N = (N_0 \ N_1)$ be an admissible operator pair (3.1). Since $\mathcal{R}(N) = \hat{\mathcal{K}}$, it follows that $\text{Ker} N^* = \{0\}$ and $\mathcal{R}(N^*)$ is a closed subspace in $H^n \oplus H^n$. Therefore there exists a Hilbert space $\hat{\mathcal{K}}^\perp$ and operators $T_j \in [H^n, \hat{\mathcal{K}}^\perp]$, $j \in \{0, 1\}$, such that the operator

$$W' = \begin{pmatrix} -N_0^* & -T_0^* \\ N_1^* & T_1^* \end{pmatrix} : \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp \rightarrow H^n \oplus H^n \quad (3.35)$$

is an isomorphism.

Next assume that W' is an isomorphism (3.35) and let $Y_{W'}(\cdot, \lambda) (\in [\hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp, H])$ be the operator solution of the equation (2.29) such that $\tilde{Y}_{W'}(0, \lambda) = W'$. Then

$$Y_{W'}(t, \lambda) = (\varphi_N(t, \lambda) \ \varphi_T(t, \lambda)) : \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp \rightarrow H, \quad \lambda \in \mathbb{C}, \quad (3.36)$$

where $\varphi_T(\cdot, \lambda) : \Delta \rightarrow [\hat{\mathcal{K}}^\perp, H]$ is the operator solution of (2.29) given by (3.27) with T in place of N . Introduce also the operator $\mathcal{J}_{W'} = (W')^{-1} J_{H^n} (W')^{-1*} (\in [\hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp])$ where J_{H^n} is the operator (2.41). Since $\mathcal{J}_{W'}^* = -\mathcal{J}_{W'}$, the operator $\mathcal{J}_{W'}$ has the block representation

$$\mathcal{J}_{W'} = \begin{pmatrix} \mathcal{J}_1 & -\mathcal{J}_2^* \\ \mathcal{J}_2 & \mathcal{J}_4 \end{pmatrix} : \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp \rightarrow \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp \quad (3.37)$$

with $\mathcal{J}_1 = -\mathcal{J}_1^*$ and $\mathcal{J}_4 = -\mathcal{J}_4^*$.

Theorem 3.11. *Assume that the following assumptions (a) are satisfied:*

(a) $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a decomposing D -triplet (2.32) for L , $N = (N_0 \ N_1)$ is an operator pair (3.1), $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ is a collection of holomorphic pairs (3.2)-(3.5), $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ is the corresponding collection (2.13) and $\Omega_\tau(\cdot)$ is the characteristic matrix (2.56).

Moreover, let W' be an isomorphism (3.35) and let $\Omega_{\tau, W'}(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp]$ be the operator function given by

$$\Omega_{\tau, W'}(\lambda) = (W')^{-1} \Omega_\tau(\lambda) (W')^{-1*}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.38)$$

Then: 1) The Green function (3.28) admits the representation

$$G_{\mathcal{P}}(x, t, \lambda) = Y_{W'}(x, \lambda) (\Omega_{\tau, W'}(\lambda) + \frac{1}{2} \text{sign}(t - x) \mathcal{J}_{W'}) Y_{W'}^*(t, \bar{\lambda}); \quad (3.39)$$

2) The operator function (3.38) has the block representation

$$\Omega_{\tau, W'}(\lambda) = \begin{pmatrix} m_{\mathcal{P}}(\lambda) & -\frac{1}{2} \mathcal{J}_2^* \\ -\frac{1}{2} \mathcal{J}_2 & 0 \end{pmatrix} : \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp \rightarrow \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}; \quad (3.40)$$

3) The equality (3.40) generates the holomorphic operator function $m_{\mathcal{P}}(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\hat{\mathcal{K}}]$ which can be also defined by the following statement:

(i) there exists a unique operator function $m_{\mathcal{P}}(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\hat{\mathcal{K}}]$ such that for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the operator function

$$v(t, \lambda) := \varphi_N(t, \lambda) (m_{\mathcal{P}}(\lambda) - \frac{1}{2} \mathcal{J}_1) - \varphi_T(t, \lambda) \mathcal{J}_2 \quad (3.41)$$

belongs to $L_2'[\hat{\mathcal{K}}, H]$ and satisfies the boundary conditions (3.18)-(3.21).

Proof. 1) The representation (3.39) is immediate from (2.43) and the obvious equality $Y_0(t, \lambda) = Y_{W'}(t, \lambda) (W')^{-1}$, $\lambda \in \mathbb{C}$.

2) Let $v_{\mathcal{P}}(\cdot, \lambda)$ be the operator function defined in Proposition 3.8 and let

$$u(x, \lambda) = (v_{\mathcal{P}}(x, \lambda) \ 0) : \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp \rightarrow H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Comparing (3.28) with (3.39) one obtains

$$u(x, \lambda) Y_{W'}^*(t, \bar{\lambda}) = Y_{W'}(x, \lambda) (\Omega_{\tau, W'}(\lambda) - \frac{1}{2} \mathcal{J}_{W'}) Y_{W'}^*(t, \bar{\lambda}), \quad x > t \quad (3.42)$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since $0 \in \rho(\tilde{Y}_{W'}(t, \bar{\lambda}))$, it follows from (3.42) that

$$u(x, \lambda) = Y_{W'}(x, \lambda) (\Omega_{\tau, W'}(\lambda) - \frac{1}{2} \mathcal{J}_{W'}), \quad x \in \Delta, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.43)$$

Next assume that the block representation of the operator function $\Omega_{\tau, W'}(\lambda)$ is

$$\Omega_{\tau, W'}(\lambda) = \begin{pmatrix} m_{\mathcal{P}}(\lambda) & \Omega_3(\lambda) \\ \Omega_2(\lambda) & \Omega_4(\lambda) \end{pmatrix} : \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp \rightarrow \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.44)$$

Then the equality (3.43) can be written as

$$(v_{\mathcal{P}}(x, \lambda) \ 0) = (\varphi_N(x, \lambda) \ \varphi_T(x, \lambda)) \begin{pmatrix} m_{\mathcal{P}}(\lambda) - \frac{1}{2} \mathcal{J}_1 & \Omega_3(\lambda) + \frac{1}{2} \mathcal{J}_2^* \\ \Omega_2(\lambda) - \frac{1}{2} \mathcal{J}_2 & \Omega_4(\lambda) - \frac{1}{2} \mathcal{J}_4 \end{pmatrix},$$

which implies the relations

$$v_{\mathcal{P}}(x, \lambda) = \varphi_N(x, \lambda) (m_{\mathcal{P}}(\lambda) - \frac{1}{2} \mathcal{J}_1) + \varphi_T(x, \lambda) (\Omega_2(\lambda) - \frac{1}{2} \mathcal{J}_2) \quad (3.45)$$

$$\Omega_3(\lambda) + \frac{1}{2} \mathcal{J}_2^* = 0, \quad \Omega_4(\lambda) - \frac{1}{2} \mathcal{J}_4 = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.46)$$

Since $\Omega_\tau(\lambda) = \Omega_\tau^*(\bar{\lambda})$, it follows from (3.38) that $\Omega_{\tau, W'}(\lambda) = \Omega_{\tau, W'}^*(\bar{\lambda})$ and by (3.44) one has $\Omega_2(\lambda) = \Omega_3^*(\bar{\lambda})$, $\Omega_4(\lambda) = \Omega_4^*(\bar{\lambda})$. Combining these relations with (3.46) and taking the equality $\mathcal{J}_4 = -\mathcal{J}_4^*$ into account one obtains

$$\Omega_3(\lambda) = -\frac{1}{2} \mathcal{J}_2^*, \quad \Omega_2(\lambda) = -\frac{1}{2} \mathcal{J}_2, \quad \Omega_4(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.47)$$

Therefore the block matrix representation (3.44) takes the form (3.40).

3) In view of (3.45) and the second equality in (3.47) the function $v(\cdot, \lambda) = v_{\mathcal{P}}(\cdot, \lambda)$ admits the representation (3.41). This and Proposition 3.8 give the statement 3). \square

Definition 3.12. The operator function $m_{\mathcal{P}}(\cdot)$ introduced in Theorem 3.11 will be called an m -function corresponding to the collection $\mathcal{P} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ or, equivalently, to the boundary value problem (3.11)-(3.15).

The m -function $m_{\mathcal{P}}(\cdot)$ will be called canonical if $\mathcal{P} \in TR^0\{\mathcal{H}_0, \mathcal{H}_1\}$ or, equivalently, if it corresponds to the canonical boundary problem (3.11), (3.16), (3.17).

Remark 3.13. Let under the conditions of Theorem 3.11 W' and \widetilde{W}' be different isomorphisms (3.35) (with the same first column), let $\Omega_{\tau, W'}(\cdot)$ and $\Omega_{\tau, \widetilde{W}'}(\cdot)$ be the corresponding functions (3.38) and let $m_{\mathcal{P}}(\lambda)$ and $\tilde{m}_{\mathcal{P}}(\lambda)$ be upper left entries in the representations (3.40). One can easily verify that $\tilde{m}_{\mathcal{P}}(\lambda) = m_{\mathcal{P}}(\lambda) + C$, $C = C^*$, which implies that the m -function $m_{\mathcal{P}}(\cdot)$ is defined by a collection $\mathcal{P} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ up to the selfadjoint constant.

For a given operator pair (3.1) introduce the operator $N' \in [H^n \oplus H^n, \hat{\mathcal{K}}]$ and the subspaces θ and θ^\perp in $H^n \oplus H^n$ by

$$N' = \begin{pmatrix} -N_0 & N_1 \end{pmatrix} : H^n \oplus H^n \rightarrow \hat{\mathcal{K}}, \quad \theta^\perp = \text{Ker} N', \quad \theta = (H^n \oplus H^n) \ominus \theta^\perp. \quad (3.48)$$

Clearly, the operator $N'_0 := N' \upharpoonright \theta$ isomorphically maps θ onto $\hat{\mathcal{K}}$ and the operator

$$\hat{N} := (N'_0)^{-1}, \quad \hat{N} \in [\hat{\mathcal{K}}, H^n \oplus H^n] \quad (3.49)$$

is the right inverse for N' , that is $N' \hat{N} = I_{\hat{\mathcal{K}}}$. Assume also that

$$\hat{N} = (\hat{N}_0 \quad \hat{N}_1)^\top : \hat{\mathcal{K}} \rightarrow H^n \oplus H^n \quad (3.50)$$

is the block matrix representation of the operator \hat{N} .

Proposition 3.14. *Let the assumptions (a) of Theorem 3.11 be satisfied and let $\alpha(\lambda) \in [\mathcal{K}_-, \hat{\mathcal{K}}]$, $\lambda \in \mathbb{C}_-$ be a linear fractional transformation of the Weyl function $M_-(\lambda)$ given by*

$$\alpha(\lambda) = (\hat{N}_1^* P_{H^n} - \hat{N}_0^* P_{H^n} M_-(\lambda)) (\tilde{D}_1(\lambda) - \tilde{D}_0(\lambda) M_-(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_-. \quad (3.51)$$

Then: 1) the m -function $m_{\mathcal{P}}(\cdot)$ is a uniformly strict Nevanlinna function satisfying the relations

$$\begin{aligned} m_{\mathcal{P}}(\mu) - m_{\mathcal{P}}^*(\lambda) &= (\mu - \bar{\lambda}) \int_0^b v_{\mathcal{P}}^*(t, \lambda) v_{\mathcal{P}}(t, \mu) dt - \alpha(\bar{\lambda}) (D_1(\bar{\lambda}) D_{01}^*(\bar{\mu}) - \\ &\quad - D_{01}(\bar{\lambda}) D_1^*(\bar{\mu}) + i D_{02}(\bar{\lambda}) D_{02}^*(\bar{\mu})) \alpha^*(\bar{\mu}), \quad \mu, \lambda \in \mathbb{C}_+ \end{aligned} \quad (3.52)$$

$$(\text{Im } \lambda)^{-1} \cdot \text{Im } (m_{\mathcal{P}}(\lambda)) \geq \int_0^b v_{\mathcal{P}}^*(t, \lambda) v_{\mathcal{P}}(t, \lambda) dt, \quad \lambda \in \mathbb{C}_+. \quad (3.53)$$

Here $D_{01}(\cdot)$ and $D_{02}(\cdot)$ are taken from (2.6) and the integral converges strongly, that is

$$\int_0^b v_{\mathcal{P}}^*(t, \lambda) v_{\mathcal{P}}(t, \mu) dt = s - \lim_{\eta \uparrow b} \int_0^\eta v_{\mathcal{P}}^*(t, \lambda) v_{\mathcal{P}}(t, \mu) dt.$$

For the canonical m -function $m_{\mathcal{P}}(\cdot)$ the identity (3.52) takes the form

$$m_{\mathcal{P}}(\mu) - m_{\mathcal{P}}^*(\lambda) = (\mu - \bar{\lambda}) \int_0^b v_{\mathcal{P}}^*(t, \lambda) v_{\mathcal{P}}(t, \mu) dt, \quad \mu, \lambda \in \mathbb{C} \setminus \mathbb{R} \quad (3.54)$$

and the inequality (3.53) turns into the equality.

2) The characteristic matrix $\Omega_\tau(\cdot)$ admits the representation

$$\Omega_\tau(\lambda) = \begin{pmatrix} \Omega_0(\lambda) & \Omega_1^* \\ \Omega_1 & \Omega_2 \end{pmatrix} : \theta \oplus \theta^\perp \rightarrow \theta \oplus \theta^\perp, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (3.55)$$

where $\Omega_2 = \Omega_2^* \in [\theta^\perp]$ and $\Omega_0(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\theta]$ is a uniformly strict Nevanlinna function associated with $m_{\mathcal{P}}(\cdot)$ by

$$\Omega_0(\lambda) = N_0'^* m_{\mathcal{P}}(\lambda) N_0' + C, \quad C = C^* \in [\theta]. \quad (3.56)$$

Moreover the following equality holds

$$m_{\mathcal{P}}(\lambda) = \hat{N}^* \Omega_\tau(\lambda) \hat{N} + \hat{C}, \quad \hat{C} = \hat{C}^* \in [\hat{\mathcal{K}}], \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.57)$$

Proof. Let W' be an isomorphism (3.35) and let $\Omega_{\tau, W'}(\lambda)$ be the operator function (3.38). Then $\Omega_{\tau}(\lambda) = W' \Omega_{\tau, W'}(\lambda) W'^*$ and the immediate calculation with taking (3.40) into account shows that

$$\Omega_{\tau}(\lambda) = N'^* m_{\mathcal{P}}(\lambda) N' + \tilde{C} \quad (3.58)$$

with some $\tilde{C} = \tilde{C}^* \in [H^n \oplus H^n]$. Multiplying the equality (3.58) by \hat{N}^* from the left and by \hat{N} from the right one obtains (3.57). Therefore $m_{\mathcal{P}}(\cdot)$ is a Nevanlinna function.

In view of (3.57) and (2.60)

$$\begin{aligned} m_{\mathcal{P}}(\mu) - m_{\mathcal{P}}^*(\lambda) &= (\mu - \bar{\lambda}) \gamma_c^*(\lambda) \gamma_c(\mu) - \alpha(\bar{\lambda}) (D_1(\bar{\lambda}) D_{01}^*(\bar{\mu}) - \\ &\quad - D_{01}(\bar{\lambda}) D_1^*(\bar{\mu}) + i D_{02}(\bar{\lambda}) D_{02}^*(\bar{\mu})) \alpha^*(\bar{\mu}), \quad \mu, \lambda \in \mathbb{C}_+, \end{aligned} \quad (3.59)$$

where

$$\gamma_c(\lambda) = \gamma_{\tau}(\lambda) \hat{N}, \quad \lambda \in \mathbb{C}_+; \quad \alpha(\lambda) = \hat{N}^* \tilde{\alpha}(\lambda), \quad \lambda \in \mathbb{C}_- \quad (3.60)$$

and $\gamma_{\tau}(\cdot)$ and $\tilde{\alpha}(\cdot)$ are the operator functions (2.58) and (2.59) respectively. Moreover by (3.59) and the inequality in (2.8) one has

$$(Im \lambda)^{-1} \cdot Im(m_{\mathcal{P}}(\lambda)) \geq \gamma_c^*(\lambda) \gamma_c(\lambda), \quad \lambda \in \mathbb{C}_+, \quad (3.61)$$

It follows from (3.2) and (3.3) that for all $\lambda \in \mathbb{C}_+$ the operator $(-\hat{C}_0(\lambda) : \hat{C}_1(\lambda))$ in (2.58) coincides with N' . This and the equality $N' \hat{N} = I_{\hat{\mathcal{K}}}$ imply that

$$\gamma_c(\lambda) = \gamma_+(\lambda) (C_0(\lambda) - C_1(\lambda) M_+(\lambda))^{-1} \upharpoonright \hat{\mathcal{K}}, \quad \lambda \in \mathbb{C}_+. \quad (3.62)$$

and, consequently, $0 \in \rho(\gamma_c^*(\lambda) \gamma_c(\lambda))$. Therefore by (3.61) the Nevanlinna function $m_{\mathcal{P}}(\cdot)$ is uniformly strict. Moreover, in view of (3.62) and (3.22) one has $(\gamma_c(\lambda) \hat{h})(t) = v_{\mathcal{P}}(t, \lambda) \hat{h}$ ($\hat{h} \in \hat{\mathcal{K}}$). Applying now Lemma 4.1, 3) from [18] to $v_{\mathcal{P}}(t, \lambda)$ we arrive at the equality

$$\gamma_c^*(\lambda) f = \int_0^b v_{\mathcal{P}}^*(t, \lambda) f(t) dt := \lim_{\eta \uparrow b} \int_0^{\eta} v_{\mathcal{P}}^*(t, \lambda) f(t) dt, \quad f = f(t) \in \mathfrak{H},$$

which implies that

$$\gamma_c^*(\lambda) \gamma_c(\mu) = \int_0^b v_{\mathcal{P}}^*(t, \lambda) v_{\mathcal{P}}(t, \mu) dt := s - \lim_{\eta \uparrow b} \int_0^{\eta} v_{\mathcal{P}}^*(t, \lambda) v_{\mathcal{P}}(t, \mu) dt. \quad (3.63)$$

Next, in view of (2.59) and (3.50) the second equality in (3.60) can be written as

$$\begin{aligned} \alpha(\lambda) &= (\hat{N}_0^* : \hat{N}_1^*) \begin{pmatrix} -P_{H^n} M_-(\lambda) \\ P_{H^n} \end{pmatrix} (\tilde{D}_1(\lambda) - \tilde{D}_0(\lambda) M_-(\lambda))^{-1} = \\ &= (\hat{N}_1^* P_{H^n} - \hat{N}_0^* P_{H^n} M_-(\lambda)) (\tilde{D}_1(\lambda) - \tilde{D}_0(\lambda) M_-(\lambda))^{-1}. \end{aligned}$$

Therefore the operator function $\alpha(\lambda)$ defined by (3.60) can be represented in the form (3.51). Combining this assertion with (3.59), (3.61) and (3.63) we obtain the identity (3.52) and the inequality (3.53). Moreover (3.52) and the equality (2.11) yield (3.54).

Finally, the equality (3.55) is immediate from (3.58) and the block representation $N' = (N'_0 \ 0) : \theta \oplus \theta^{\perp} \rightarrow \hat{\mathcal{K}}$. \square

Corollary 3.15. *Let the assumptions (a) of Theorem 3.11 be satisfied. Then the following statements are equivalent:*

- (i) *the characteristic matrix $\Omega_\tau(\cdot)$ is a uniformly strict Nevanlinna function;*
- (ii) *the operator $N = \begin{pmatrix} N_0 & N_1 \end{pmatrix}$ (3.1) isomorphically maps $H^n \oplus H^n$ onto $\hat{\mathcal{K}}$.*

If in addition $\dim H < \infty$, then the statement (i) is equivalent to the following one:

- (iii) *the operator L_0 has maximal deficiency indices $n_+(L_0) = n_-(L_0) = 2n \dim H$, $\mathcal{H}'_0 = \mathcal{H}'_1 =: \mathcal{H}'$ (i.e., $\Pi = \{H^n \oplus \mathcal{H}', \Gamma_0, \Gamma_1\}$ is a decomposing boundary triplet for L), $\dim \mathcal{H}' = n \dim H$ and the collection \mathcal{P} can be represented as the holomorphic Nevanlinna pair (c.f. Remark 2.4, 2)) $C(\lambda) = \begin{pmatrix} C_0(\lambda) & C_1(\lambda) \end{pmatrix}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$,*

$$C_0(\lambda) = \begin{pmatrix} N_0 & C'_0(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}' \rightarrow \hat{\mathcal{K}}, \quad C_1(\lambda) = \begin{pmatrix} N_1 & C'_1(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}' \rightarrow \hat{\mathcal{K}}, \quad (3.64)$$

where $\dim \hat{\mathcal{K}} = 2n \dim H$ and the operator $N = \begin{pmatrix} N_0 & N_1 \end{pmatrix} : H^n \oplus H^n \rightarrow \hat{\mathcal{K}}$ is an isomorphism.

Proof. It follows from (3.55) that the Nevanlinna function $\Omega_\tau(\cdot)$ is uniformly strict if and only if $\theta^\perp = \{0\}$. Moreover by (3.48) one has $\theta^\perp = \{0\} \Leftrightarrow \text{Ker} N (= \text{Ker} N') = \{0\}$. This yields the equivalence (i) \Leftrightarrow (ii).

Next assume that $\dim H < \infty$ and prove the equivalence (ii) \Leftrightarrow (iii). If $0 \in \rho(N)$, then $\dim \hat{\mathcal{K}} = \dim(H^n \oplus H^n) = 2n \dim H$ and by (3.10) $n_-(L_0) = n_+(L_0) = 2n \dim H$. This and the second relation in (2.35) imply that $\dim \mathcal{H}_1 = \dim \mathcal{H}_0 = 2n \dim H$ and hence $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$. Therefore $\mathcal{H}'_0 = \mathcal{H}'_1$ and $\Pi = \{H^n \oplus \mathcal{H}', \Gamma_0, \Gamma_1\}$ is a decomposing boundary triplet for L . Moreover by (2.15) the Hilbert spaces $\hat{\mathcal{K}} \oplus \mathcal{K}'_j$ in (3.2)-(3.5) satisfy the equalities $\dim(\hat{\mathcal{K}} \oplus \mathcal{K}'_j) = \dim \mathcal{H} = 2n \dim H = \dim \hat{\mathcal{K}}$. Hence $\mathcal{K}'_j = \{0\}$, $j \in \{0, 1\}$ and the equalities (3.2)-(3.5) take the form (3.64), which yields the implication (ii) \Rightarrow (iii). The inverse implication (iii) \Rightarrow (ii) is obvious. Thus in the case $\dim H < \infty$ the equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii) hold. \square

Remark 3.16. 1) It follows from Corollary 3.15 that in the case $\dim H < \infty$ and $n_-(L_0) < 2n \dim H$ the characteristic matrix $\Omega_\tau(\cdot)$ corresponding to the boundary operators (3.2)-(3.5) is not uniformly strict Nevanlinna function. In particular, by Proposition 3.2 this statement holds for each canonical characteristic matrix corresponding to the constant Nevanlinna collection (2.10).

2) Let $\mathcal{P}_0 \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ be the collection (3.2)-(3.5) with $\hat{\mathcal{K}} = H^n$, $\mathcal{K}'_j = \mathcal{H}'_j$, $j \in \{0, 1\}$ and $C_0(\lambda) = I_{\mathcal{H}_0}$, $C_1(\lambda) = 0_{\mathcal{H}_1, \mathcal{H}_0}$, $D_0 = P_1$ and $D_1 = 0_{\mathcal{H}_1}$. Then the corresponding m -function $m_{\mathcal{P}_0}(\cdot)$ coincides with the operator function $m(\cdot)$ defined by (2.39) and (2.40). Note in this connection that the statements of Theorem 3.11 and Proposition 3.14 for $m(\lambda) (= m_{\mathcal{P}_0}(\lambda))$ were obtained in our paper [17].

3.4. m -function and a characteristic matrix as the Weyl functions. In this subsection we show that a canonical m -function $m_{\mathcal{P}}(\cdot)$ is the Weyl function of some symmetric extension $\hat{A} \in \text{Ext}_{L_0}$, while a canonical characteristic matrix $\Omega_\tau(\cdot)$ is the Weyl function of the minimal operator L_0 (the last statement holds under some additional assumptions).

Proposition 3.17. *Assume that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a decomposing boundary triplet (2.32) for L (with $\mathcal{H} = H^n \oplus \mathcal{H}'$) and $\mathcal{P} = \{C_0, C_1\} \in TR^0(\mathcal{H})$ is an operator pair defined by (3.7) with $\mathcal{H}'_0 = \mathcal{H}'_1 =: \mathcal{H}'$. Moreover let $X_j \in [\mathcal{H}, \mathcal{K}]$, $j \in \{0, 1\}$ be operators such that the operator*

$$X = \begin{pmatrix} C_0 & -C_1 \\ X_0 & -X_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{K} \oplus \mathcal{K} \quad (3.65)$$

satisfies the relations $X^ J_{\mathcal{K}} X = J_{\mathcal{H}}$ and $0 \in \rho(X)$ (such operators X_0 and X_1 exist, because $\{(C_0, -C_1); \mathcal{K}\}$ is a selfadjoint operator pair). Suppose also that*

$$X_0 = \begin{pmatrix} X_{01} & X_{02} \\ X_{03} & X_{04} \end{pmatrix} : H^n \oplus \mathcal{H}' \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}', \quad X_1 = \begin{pmatrix} X_{11} & X_{12} \\ X_{13} & X_{14} \end{pmatrix} : H^n \oplus \mathcal{H}' \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}' \quad (3.66)$$

are the block matrix representations of X_j , $j \in \{0, 1\}$ and

$$\tilde{C}'_0 := \begin{pmatrix} C'_{01} \\ C'_{02} \\ X_{02} \end{pmatrix} : \mathcal{H}' \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}' \oplus \hat{\mathcal{K}}, \quad \tilde{C}'_1 := \begin{pmatrix} C'_{11} \\ C'_{12} \\ X_{12} \end{pmatrix} : \mathcal{H}' \rightarrow \hat{\mathcal{K}} \oplus \mathcal{K}' \oplus \hat{\mathcal{K}}. \quad (3.67)$$

Then: 1) the operator $\hat{A} := L \upharpoonright \mathcal{D}(\hat{A})$ defined by the decomposing boundary conditions

$$\mathcal{D}(\hat{A}) = \{y \in \mathcal{D} : y^{(1)}(0) = y^{(2)}(0) = 0, \tilde{C}'_0 \Gamma'_0 y - \tilde{C}'_1 \Gamma'_1 y = 0\} \quad (3.68)$$

is a closed symmetric extension of L_0 ;

2) the adjoint \hat{A}^ of \hat{A} is defined by the boundary condition*

$$\mathcal{D}(\hat{A}^*) = \{y \in \mathcal{D} : C'_{02} \Gamma'_0 y - C'_{12} \Gamma'_1 y = 0\}; \quad (3.69)$$

3) the maps $\hat{\Gamma}_j : \mathcal{D}(\hat{A}^) \rightarrow \hat{\mathcal{K}}$, $j \in \{0, 1\}$ given by*

$$\hat{\Gamma}_0 y = N_0 y^{(2)}(0) + N_1 y^{(1)}(0) + C'_{01} \Gamma'_0 y - C'_{11} \Gamma'_1 y, \quad (3.70)$$

$$\hat{\Gamma}_1 y = X_{01} y^{(2)}(0) + X_{11} y^{(1)}(0) + X_{02} \Gamma'_0 y - X_{12} \Gamma'_1 y, \quad y \in \mathcal{D}(\hat{A}^*) \quad (3.71)$$

form a boundary triplet $\hat{\Pi} = \{\hat{\mathcal{K}}, \hat{\Gamma}_0, \hat{\Gamma}_1\}$ for \hat{A}^ ;*

4) the corresponding γ -field and Weyl function (2.25) for $\hat{\Pi}$ are

$$(\hat{\gamma}(\lambda)h)(t) = v_{\mathcal{P}}(t, \lambda)h, \quad h \in \hat{\mathcal{K}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \quad (3.72)$$

$$\hat{M}(\lambda) = m_{\mathcal{P}}(\lambda) + \hat{D}, \quad \hat{D} = \hat{D}^* \in [\hat{\mathcal{K}}], \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.73)$$

Proof. According to [5] the operators

$$\tilde{\Gamma}_0 = C_0 \Gamma_0 - C_1 \Gamma_1, \quad \tilde{\Gamma}_1 = X_0 \Gamma_0 - X_1 \Gamma_1 \quad (3.74)$$

form a boundary triplet $\tilde{\Pi} = \{\hat{\mathcal{K}} \oplus \mathcal{K}', \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ for L and the γ -field for $\tilde{\Pi}$ is

$$\tilde{\gamma}(\lambda) = \gamma(\lambda)(C_0 - C_1 M(\lambda))^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (3.75)$$

where $\gamma(\cdot)$ and $M(\cdot)$ are the γ -field and the Weyl function for Π respectively. Moreover by (3.7), (3.66) and (2.32) the equalities (3.74) can be written as

$$\tilde{\Gamma}_0 y = \{N_0 y^{(2)}(0) + N_1 y^{(1)}(0) + C'_{01} \Gamma'_0 y - C'_{11} \Gamma'_1 y, \quad C'_{02} \Gamma'_0 y - C'_{12} \Gamma'_1 y\} \quad (3.76)$$

$$\begin{aligned} \tilde{\Gamma}_1 y = \{ & X_{01} y^{(2)}(0) + X_{11} y^{(1)}(0) + X_{02} \Gamma'_0 y - X_{12} \Gamma'_1 y, \\ & X_{03} y^{(2)}(0) + X_{13} y^{(1)}(0) + X_{04} \Gamma'_0 y - X_{14} \Gamma'_1 y\}, \quad y \in \mathcal{D} \end{aligned} \quad (3.77)$$

Applying Proposition 4.1 from [5] to the triplet $\tilde{\Pi}$ one obtains the following assertions:

- (i) the equality $\mathcal{D}(\hat{A}) = \{y \in \mathcal{D} : \tilde{\Gamma}_0 y = P_{\hat{\mathcal{K}}} \tilde{\Gamma}_1 y = 0\}$ defines a symmetric extension $\hat{A} \in \text{Ext}_{L_0}$;
- (ii) the adjoint \hat{A}^* of \hat{A} is given by $\mathcal{D}(\hat{A}^*) = \{y \in \mathcal{D} : P_{\hat{\mathcal{K}}} \tilde{\Gamma}_0 y = 0\}$;
- (iii) the operators $\hat{\Gamma}_j : \mathcal{D}(\hat{A}^*) \rightarrow \hat{\mathcal{K}}, j \in \{0, 1\}$ given by

$$\hat{\Gamma}_0 y = \tilde{\Gamma}_0 y, \quad \hat{\Gamma}_1 y = P_{\hat{\mathcal{K}}} \tilde{\Gamma}_1 y, \quad y \in \mathcal{D}(\hat{A}^*) \quad (3.78)$$

form a boundary triplet $\hat{\Pi} = \{\hat{\mathcal{K}}, \hat{\Gamma}_0, \hat{\Gamma}_1\}$ for \hat{A}^* and the γ -field for $\hat{\Pi}$ is

$$\hat{\gamma}(\lambda) = \tilde{\gamma}(\lambda) \upharpoonright \hat{\mathcal{K}} = \gamma(\lambda)(C_0 - C_1 M(\lambda))^{-1} \upharpoonright \hat{\mathcal{K}}. \quad (3.79)$$

Next we show that the operator \hat{A} and the triplet $\hat{\Pi}$ satisfy the statement 1)-4) of the proposition.

First observe that (3.69) is immediate from the assertion (ii) and the equality (3.76). Next, (3.76), (3.77) and the assertion (i) yield the equivalence

$$y \in \mathcal{D}(\hat{A}) \iff \begin{cases} N_0 y^{(2)}(0) + N_1 y^{(1)}(0) + C'_{01} \Gamma'_0 y - C'_{11} \Gamma'_1 y = 0 \\ C'_{02} \Gamma'_0 y - C'_{12} \Gamma'_1 y = 0 \\ X_{01} y^{(2)}(0) + X_{11} y^{(1)}(0) + X_{02} \Gamma'_0 y - X_{12} \Gamma'_1 y = 0 \end{cases}. \quad (3.80)$$

Let \mathcal{D}'_2 be the set of all functions $y \in \mathcal{D}$ finite at the point b . According to [17] $\Gamma'_0 \upharpoonright \mathcal{D}'_2 = \Gamma'_1 \upharpoonright \mathcal{D}'_2 = 0$ and in view of (3.69) $\mathcal{D}'_2 \subset \mathcal{D}(\hat{A}^*)$. This and the Lagrange's identity (2.30) imply that $y^{(1)}(0) = y^{(2)}(0) = 0$ for all $y \in \mathcal{D}(\hat{A})$. Combining these equalities with (3.80) and taking (3.67) into account we arrive at (3.68).

The statement 3) of the proposition follows from (3.78) and the equalities (3.76), (3.77). Moreover combining (3.79) with (2.38) and (3.22) one obtains the equality (3.72) for $\hat{\gamma}(\lambda)$. Finally to prove (3.73) note that the Weyl function $\hat{M}(\cdot)$ for $\hat{\Pi}$ satisfies the identity [4]

$$\hat{M}(\mu) - \hat{M}^*(\lambda) = (\mu - \bar{\lambda}) \hat{\gamma}^*(\lambda) \hat{\gamma}(\mu), \quad \mu, \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.81)$$

Moreover in view of (3.79) $\hat{\gamma}(\lambda)$ coincides with the operator function $\gamma_c(\lambda)$ defined by (3.62). Therefore the equality (3.63) holds with $\hat{\gamma}(\lambda)$ in place of $\gamma_c(\lambda)$ and by (3.54)

$$m_{\mathcal{P}}(\mu) - m_{\mathcal{P}}^*(\lambda) = (\mu - \bar{\lambda}) \hat{\gamma}^*(\lambda) \hat{\gamma}(\mu), \quad \mu, \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.82)$$

Now comparing (3.81) and (3.82) we arrive at the relation (3.73). \square

Corollary 3.18. *Assume that $\Pi = \{H^n \oplus \mathcal{H}', \Gamma_0, \Gamma_1\}$ is a decomposing boundary triplet for L , $N = (N_0 \ N_1)$ is an operator pair (3.1) such that $0 \in \rho(N)$, N' is the operator (3.48), $\mathcal{P} = \{C_0, C_1\} \in TR^0(\mathcal{H})$ is the operator pair given by (c. f. (3.64))*

$$C_0 = (N_0 \ C'_0) : H^n \oplus \mathcal{H}' \rightarrow \hat{\mathcal{K}}, \quad C_1 = (N_1 \ C'_1) : H^n \oplus \mathcal{H}' \rightarrow \hat{\mathcal{K}} \quad (3.83)$$

and

$$X = \begin{pmatrix} (N_0 & C'_0) & (-N_1 & -C'_1) \\ (X_{01} & X_{02}) & (-X_{11} & -X_{12}) \end{pmatrix} : (H^n \oplus \mathcal{H}') \oplus (H^n \oplus \mathcal{H}') \rightarrow \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}$$

is the operator satisfying the relations $X^* J_{\hat{\mathcal{K}}} X = J_{\mathcal{H}}$ and $0 \in \rho(X)$. Moreover let $\Omega_\tau(\cdot)$ be the canonical characteristic matrix corresponding to $\tau = \{(C_0, C_1); \hat{\mathcal{K}}\}$. Then the operators

$$\Gamma_0^\Omega y = \{-y^{(2)}(0), y^{(1)}(0)\} + N'^{-1} C'_0 \Gamma'_0 y - N'^{-1} C'_1 \Gamma'_1 y \in H^n \oplus H^n \quad (3.84)$$

$$\Gamma_1^\Omega y = N'^*(X_{01} y^{(2)}(0) + X_{11} y^{(1)}(0) + X_{02} \Gamma'_0 y - X_{12} \Gamma'_1 y) \in H^n \oplus H^n, \quad y \in \mathcal{D} \quad (3.85)$$

form a boundary triplet $\{H^n \oplus H^n, \Gamma_0^\Omega, \Gamma_1^\Omega\}$ for L with the corresponding Weyl function

$$M_\Omega(\lambda) = \Omega_\tau(\lambda) + \tilde{D}, \quad \tilde{D} = \tilde{D}^* \in [H^n \oplus H^n]. \quad (3.86)$$

Proof. Comparing (3.7) with (3.83) one obtains $\mathcal{K}' = \{0\}$ and hence $C'_{02} = C'_{12} = 0$. Therefore the extensions (3.69) and (3.68) take the form $\hat{A}^* = L$, $\hat{A} = L_0$ and by Proposition 3.17 the operators (3.70) and (3.71) form the boundary triplet $\hat{\Pi} = \{\hat{\mathcal{K}}, \hat{\Gamma}_0, \hat{\Gamma}_1\}$ for L with the Weyl function (3.73). Next, according to Proposition 3.13 in [5] the collection $\{H^n \oplus H^n, \Gamma_0^\Omega, \Gamma_1^\Omega\}$ with $\Gamma_0^\Omega = N'^{-1} \hat{\Gamma}_0$ and $\Gamma_1^\Omega = N'^* \hat{\Gamma}_1$ is a boundary triplet for L and the corresponding Weyl function is

$$M_\Omega(\lambda) = N'^* \hat{M}(\lambda) N' \quad (3.87)$$

Now the equalities (3.84) and (3.85) are implied by (3.70), (3.71) and the relation

$$N'^{-1}(N_0 y^{(2)}(0) + N_1 y^{(1)}(0)) = N'^{-1} N' \{-y^{(2)}(0), y^{(1)}(0)\} = \{-y^{(2)}(0), y^{(1)}(0)\}.$$

Moreover since $0 \in \rho(N') (\Leftrightarrow 0 \in \rho(N))$, it follows from (3.48) that $\theta^\perp = \{0\}$ and by (3.55), (3.56) one has

$$\Omega_\tau(\lambda) = \Omega_0(\lambda) = N'^* m_{\mathcal{P}}(\lambda) N' + C, \quad C = C^* \in [H^n \oplus H^n].$$

Combining this equality with (3.73) and (3.87) one obtains the relation (3.86) for $M_\Omega(\lambda)$. \square

Remark 3.19. It follows from Corollary 3.15 that in the case $\dim H < \infty$ the conditions of Corollary 3.18 are necessary (and, by the statement of this corollary, sufficient) for the canonical characteristic matrix $\Omega_\tau(\cdot)$ be the Weyl function of the operator L_0 .

4. SPECTRAL FUNCTIONS OF DIFFERENTIAL OPERATORS

4.1. **The space $L_2(\Sigma; \mathcal{H})$.** Let \mathcal{H} be a separable Hilbert space.

Definition 4.1. A nondecreasing operator function $\Sigma : \mathbb{R} \rightarrow [\mathcal{H}]$ is called a distribution if it is strongly left continuous and satisfies the equality $\Sigma(0) = 0$.

Let $\Sigma : \mathbb{R} \rightarrow [\mathcal{H}]$ be a distribution and let $f(\cdot)$, $g(\cdot)$ be vector functions defined on the segment $[\alpha, \beta]$ with values in \mathcal{H} . Consider the Riemann-Stieltjes integral [1]

$$\int_{\alpha}^{\beta} (d\Sigma(t)f(t), g(t)) = \lim_{d_{\pi} \rightarrow 0} \sum_{k=1}^n ((\Sigma(t_k) - \Sigma(t_{k-1}))f(\xi_k), g(\xi_k)), \quad (4.1)$$

where $\pi = \{\alpha = t_0 < t_1 < \dots < t_n = \beta\}$ is a partition of $[\alpha, \beta]$, $\xi_k \in [t_{k-1}, t_k]$ and d_{π} is the diameter of π . As is known (see for instance [13]) in the case $\dim \mathcal{H} = \infty$ there exist a distribution $\Sigma(\cdot)$ and continuous functions $f(\cdot)$ and $g(\cdot)$ for which the integral (4.1) does not exist. At the same time holomorphy of $f(\cdot)$ and $g(\cdot)$ on the segment $[\alpha, \beta]$ is a sufficient condition for existence of such an integral [23].

Definition 4.2. A function $f : [\alpha, \beta] \rightarrow \mathcal{H}$ will be called piecewise holomorphic if there is a partition $\alpha = t_0 < t_1 < \dots < t_n = \beta$ such that each restriction $f \upharpoonright [t_{k-1}, t_k]$ admits a holomorphic continuation $\tilde{f}_k(\cdot)$ on some interval $(\tilde{t}_{k-1}, \tilde{t}_k) \supset [t_{k-1}, t_k]$.

A function $f : \mathbb{R} \rightarrow \mathcal{H}$ will be called piecewise holomorphic if it is so on each finite half-interval $[\alpha, \beta]$.

It follows from Definition 4.2 that a piecewise holomorphic function is strongly right continuous.

Let $\Sigma : \mathbb{R} \rightarrow [\mathcal{H}]$ be a distribution and let $f, g : [\alpha, \beta] \rightarrow \mathcal{H}$ be a pair of piecewise holomorphic functions. It is clear that there exists a partition of $[\alpha, \beta]$ satisfying the conditions of Definition 4.2 for both functions $f(\cdot)$ and $g(\cdot)$. By using such a partition introduce the integral

$$\int_{[\alpha, \beta]} (d\Sigma(t)f(t), g(t)) = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} (d\Sigma(t)\tilde{f}_k(t), \tilde{g}_k(t)). \quad (4.2)$$

Note that for a pair of continuous functions $f, g : [\alpha, \beta] \rightarrow \mathcal{H}$ piecewise holomorphic on $[\alpha, \beta]$ there exists the integral (4.1) which coincides with that of (4.2).

For a given distribution $\Sigma : \mathbb{R} \rightarrow [\mathcal{H}]$ denote by $Hol(\Sigma, \mathcal{H})$ the set of all piecewise holomorphic functions $f : \mathbb{R} \rightarrow \mathcal{H}$ such that

$$\int_{\mathbb{R}} (d\Sigma(t)f(t), f(t)) := \lim_{[\alpha, \beta] \rightarrow \mathbb{R}} \int_{[\alpha, \beta]} (d\Sigma(t)f(t), f(t)) < \infty.$$

One can easily prove that for each pair $f, g \in Hol(\Sigma, \mathcal{H})$ there exists the integral

$$(f, g)_{Hol(\Sigma, \mathcal{H})} = \int_{\mathbb{R}} (d\Sigma(t)f(t), g(t)) := \lim_{[\alpha, \beta] \rightarrow \mathbb{R}} \int_{[\alpha, \beta]} (d\Sigma(t)f(t), g(t)). \quad (4.3)$$

This implies that $Hol(\Sigma, \mathcal{H})$ is a linear space with the semi-definite scalar product (4.3).

Next recall the definition of the space $L_2(\Sigma; H)$ as it was given in the book [1].

A function $f : \mathbb{R} \rightarrow \mathcal{H}$ is called finite-dimensional if there is a subspace $\mathcal{H}_f \subset \mathcal{H}$ such that $\dim \mathcal{H}_f < \infty$ and $f(t) \in \mathcal{H}_f$, $t \in \mathbb{R}$. For a given distribution $\Sigma : \mathbb{R} \rightarrow [\mathcal{H}]$ denote by $C_{00}(\mathcal{H})$ the linear space of all strongly continuous finite-dimensional functions $f : \mathbb{R} \rightarrow \mathcal{H}$ with compact support $\text{supp } f$. Clearly the equality

$$(f, g)_{L_2(\Sigma; \mathcal{H})} = \int_{\mathbb{R}} (d\Sigma(t)f(t), g(t)) := \int_{\alpha}^{\beta} (d\Sigma(t)f(t), g(t)), \quad f, g \in C_{00}(\mathcal{H}) \quad (4.4)$$

with $[\alpha, \beta] \supset (\text{supp } f \cup \text{supp } g)$ defines the semi-definite scalar product on $C_{00}(\mathcal{H})$. The completion of $C_{00}(\mathcal{H})$ with respect to this product is a semi-Hilbert space $\tilde{L}_2(\Sigma; \mathcal{H})$. The quotient of $\tilde{L}_2(\Sigma; \mathcal{H})$ over the kernel $\{f \in \tilde{L}_2(\Sigma; \mathcal{H}) : (f, f)_{L_2(\Sigma; \mathcal{H})} = 0\}$ is the Hilbert space $L_2(\Sigma; H)$.

Denote by $Hol_0(\Sigma, \mathcal{H})$ the set of all strongly continuous, piecewise holomorphic and finite dimensional functions $f : \mathbb{R} \rightarrow \mathcal{H}$ with a compact support. It is clear that $Hol_0(\Sigma, \mathcal{H}) = Hol(\Sigma, \mathcal{H}) \cap C_{00}(\mathcal{H})$ and consequently $Hol_0(\Sigma, \mathcal{H})$ is a linear manifold both in $Hol(\Sigma, \mathcal{H})$ and $C_{00}(\mathcal{H})$. Moreover the semiscalar products (4.3) and (4.4) coincide on $Hol_0(\Sigma, \mathcal{H})$.

By using the Taylor expansions of the function $f \in Hol(\Sigma, \mathcal{H})$ one can prove the following proposition.

Proposition 4.3. *The set $Hol_0(\Sigma, \mathcal{H})$ is a dense linear manifold both in $Hol(\Sigma, \mathcal{H})$ and $C_{00}(\mathcal{H})$, which implies that the closure of $Hol(\Sigma, \mathcal{H})$ coincides with $\tilde{L}_2(\Sigma; \mathcal{H})$. In other words the semi-Hilbert space $\tilde{L}_2(\Sigma; \mathcal{H})$ can be considered as the completion of $Hol(\Sigma, \mathcal{H})$.*

Remark 4.4. In connection with Proposition 4.3 note that the intrinsic functional description of the spaces $\tilde{L}_2(\Sigma; \mathcal{H})$ and $L_2(\Sigma; \mathcal{H})$ in the case $\dim \mathcal{H} < \infty$ was obtained in [11]. Moreover in the case $\dim \mathcal{H} = \infty$ the description of these spaces in terms of the direct integrals of Hilbert spaces can be found in the recent paper [13].

4.2. Spectral functions. Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D -triplet (2.32) for L and let $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ be a Nevanlinna collection defined by (2.13) and (2.44), (2.45). For this collection consider the boundary problem (2.46)-(2.48). According to Remark 2.12 this problem defines the spectral function $F_\tau(t)$ of the operator L_0 .

Next assume that $\tilde{\mathcal{K}}$ is a separable Hilbert space and $\varphi(\cdot, \lambda) : \Delta \rightarrow [\tilde{\mathcal{K}}, H]$ is an operator solution of the equation (2.29) with the constant initial data $\tilde{\varphi}(0, \lambda) \equiv \tilde{\varphi}_0(\in [\tilde{\mathcal{K}}, H^{2n}])$, $\lambda \in \mathbb{C}$, such that $0 \in \hat{\rho}(\tilde{\varphi}_0)$. Denote by \mathfrak{H}_0 the set of all functions $f \in \mathfrak{H}(= L_2(\Delta; H))$ with $\text{supp } f \subset [0, \beta]$ ($\beta < b$ depends on f) and consider the Fourier transform $g_f : \mathbb{R} \rightarrow \tilde{\mathcal{K}}$ of a function $f \in \mathfrak{H}_0$ given by

$$g_f(s) = \int_0^b \varphi^*(t, s) f(t) dt. \quad (4.5)$$

Definition 4.5. A distribution $\Sigma(\cdot) = \Sigma_{\tau, \varphi}(\cdot) : \mathbb{R} \rightarrow [\tilde{\mathcal{K}}]$ is called a spectral function of the boundary problem (2.46)-(2.48) corresponding to the solution $\varphi(\cdot, \lambda)$ if for each

function $f \in \mathfrak{H}_0$ the Fourier transform (4.5) satisfies the equality

$$((F_\tau(\beta) - F_\tau(\alpha))f, f)_{\mathfrak{H}} = \int_{[\alpha, \beta]} (d\Sigma_{\tau, \varphi}(s)g_f(s), g_f(s)), \quad [\alpha, \beta] \subset \mathbb{R}. \quad (4.6)$$

Note that the integral in the right hand part of (4.6) exists, because the function $g_f(\cdot)$ is holomorphic on \mathbb{R} . Moreover by (4.6) $g_f(\cdot) \in \text{Hol}(\Sigma_{\tau, \varphi}, \tilde{\mathcal{K}})$ and the following Parseval equality holds

$$(\|f\|_{\mathfrak{H}}^2 =) \int_0^b \|f(t)\|_H^2 dt = \int_{\mathbb{R}} (d\Sigma_{\tau, \varphi}(s)g_f(s), g_f(s)) (= \|g_f\|_{L_2(\Sigma_{\tau, \varphi}; \tilde{\mathcal{K}})}^2), \quad f \in \mathfrak{H}_0.$$

This implies that the linear operator $V : \mathfrak{H} \rightarrow L_2(\Sigma_{\tau, \varphi}; \tilde{\mathcal{K}})$ defined on the dense linear manifold $\mathfrak{H}_0 \subset \mathfrak{H}$ by $(Vf)(s) = g_f(s)$ is an isometry.

Definition 4.6. A spectral function $\Sigma_{\tau, \varphi}(\cdot)$ is called orthogonal if $V\mathfrak{H} = L_2(\Sigma_{\tau, \varphi}; \tilde{\mathcal{K}})$ or equivalently if the set of all Fourier transforms $\{g_f(\cdot) : f \in \mathfrak{H}_0\}$ is dense in $L_2(\Sigma_{\tau, \varphi}; \tilde{\mathcal{K}})$.

Theorem 4.7. Let \tilde{H} be a Hilbert space with $\dim \tilde{H} = 2n \cdot \dim H$, let $W \in [\tilde{H}, H^{2n}]$ be an isomorphism and let $Y_W(\cdot, \lambda) : \Delta \rightarrow [\tilde{H}, H]$ be an operator solution of the equation (2.29) with the initial data $\tilde{Y}_W(0, \lambda) = W$, $\lambda \in \mathbb{C}$. Then for each collection $\tau \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ there exists the unique spectral function $\Sigma_{\tau, W} : \mathbb{R} \rightarrow [\tilde{H}]$ of the boundary problem (2.46)-(2.48) corresponding to the solution $Y_W(\cdot, \lambda)$. This function is defined by the equality

$$\Sigma_{\tau, W}(s) = s - \lim_{\delta \rightarrow +0} w - \lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \text{Im} \Omega_{\tau, W}(\sigma + i\varepsilon) d\sigma, \quad (4.7)$$

where $\Omega_{\tau, W} : \mathbb{C} \setminus \mathbb{R} \rightarrow [\tilde{H}]$ is a Nevanlinna operator function given by

$$\Omega_{\tau, W}(\lambda) = W^{-1} \Omega_{\tau}(\lambda) W^{-1*}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (4.8)$$

Moreover the spectral function $\Sigma_{\tau, W}(\cdot)$ is orthogonal if and only if $\tau \in \tilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$.

One can prove Theorem 4.7 by using the Stieltjes -Livšic formula [12, 23] in the same way as in [24] (the scalar case $\dim H = 1$) and [3] (the case $\dim H \leq \infty$). Moreover in the scalar case other methods of the proof can be found in the books [20, 7].

Theorem 4.8. Assume that under the conditions of Theorem 4.7 $\Sigma_{\tau, W}(\cdot)$ is a spectral function of the boundary problem (2.46)-(2.48) and $V : \mathfrak{H} \rightarrow L_2(\Sigma_{\tau, W}; \tilde{H})$ is the corresponding isometry given by the Fourier transform (4.5) with $\varphi(t, s) = Y_W(t, s)$. Moreover, let $\text{Hol}^0(\tilde{H})$ be a linear manifold of all piecewise holomorphic functions $g : \mathbb{R} \rightarrow \tilde{H}$ with compact support. Then $\text{Hol}^0(\tilde{H})$ is dense in $L_2(\Sigma_{\tau, W}; \tilde{H})$ and

$$(V^*g)(t) = \int_{\mathbb{R}} Y_W(t, s) d\Sigma_{\tau, W}(s)g(s), \quad g = g(s) \in \text{Hol}^0(\tilde{H}), \quad (4.9)$$

where $V^* : L_2(\Sigma_{\tau,W}; \tilde{H}) \rightarrow \mathfrak{H}$ is the adjoint operator and the integral is understood as the sum of integrals similarly to (4.2). In particular formula (4.9) implies that the inverse Fourier transform is

$$f(t) = \int_{\mathbb{R}} Y_W(t, s) d\Sigma_{\tau,W}(s) g_f(s). \quad (4.10)$$

In the case $\dim H < \infty$ the proof of Theorem 4.8 can be found in [20, 7, 24]. In the case $\dim H = \infty$ a somewhat weaker result (only the inverse transform (4.10)) is contained in [3]. In this connection note that in the case $\dim H = \infty$ the piecewise holomorphy of a function $g(\cdot)$ is essential, because otherwise the integral in (4.9) may not exist.

Our next goal is to obtain a description of all spectral functions $\Sigma_{\tau,W}(\cdot)$ immediately in terms of a boundary parameter τ . Namely, using the block representations (2.39) and (2.40) of the Weyl functions $M_{\pm}(\cdot)$ introduce the operator functions $\Omega_{\tau_0}(\lambda) (\in [H^{2n}])$, $S_+(\lambda) (\in [\mathcal{H}_0, H^{2n}])$ and $S_-(z) (\in [\mathcal{H}_1, H^{2n}])$ by setting

$$\Omega_{\tau_0}(\lambda) = \begin{pmatrix} m(\lambda) & -\frac{1}{2}I_{H^n} \\ -\frac{1}{2}I_{H^n} & 0 \end{pmatrix} : H^n \oplus H^n \rightarrow H^n \oplus H^n, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \quad (4.11)$$

$$S_+(\lambda) = \begin{pmatrix} -m(\lambda) & -M_{2+}(\lambda) \\ I_{H^n} & 0 \end{pmatrix} : H^n \oplus \mathcal{H}'_0 \rightarrow H^n \oplus H^n, \quad \lambda \in \mathbb{C}_+ \quad (4.12)$$

$$S_-(z) = \begin{pmatrix} -m(z) & -M_{2-}(z) \\ I_{H^n} & 0 \end{pmatrix} : H^n \oplus \mathcal{H}'_1 \rightarrow H^n \oplus H^n, \quad z \in \mathbb{C}_-. \quad (4.13)$$

Note that $\Omega_{\tau_0}(\lambda)$ is a characteristic matrix corresponding to the collection $\tau_0 = \{\tau_{0+}, \tau_{0-}\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ with $\tau_{0+} = \{0\} \oplus \mathcal{H}_1 (\in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1))$.

Theorem 4.9. *Let the assumptions of Theorem 4.7 be satisfied and let $\Omega_{\tau_0,W}(\lambda) (\in [\tilde{H}])$, $S_{W,+}(\lambda) (\in [\mathcal{H}_0, \tilde{H}])$ and $S_{W,-}(z) (\in [\mathcal{H}_1, \tilde{H}])$ be the operator functions given by*

$$\begin{aligned} \Omega_{\tau_0,W}(\lambda) &= W^{-1}\Omega_{\tau_0}(\lambda)W^{-1*}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}; \\ S_{W,+}(\lambda) &= W^{-1}S_+(\lambda), \quad \lambda \in \mathbb{C}_+; \quad S_{W,-}(z) = W^{-1}S_-(z), \quad z \in \mathbb{C}_-. \end{aligned}$$

Then for each collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ the equality

$$\Omega_{\tau,W}(\lambda) = \Omega_{\tau_0,W}(\lambda) - S_{W,+}(\lambda)(\tau_+(\lambda) + M_+(\lambda))^{-1}S_{W,-}^*(\bar{\lambda}), \quad \lambda \in \mathbb{C}_+ \quad (4.14)$$

together with (4.7) defines a (unique) spectral function $\Sigma_{\tau,W}(\cdot)$ of the boundary problem (2.46)-(2.48) corresponding to the solution $Y_W(\cdot, \lambda)$. Moreover a spectral function $\Sigma_{\tau,W}(\cdot)$ is orthogonal if and only if $\tau \in \tilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$.

Proof. According to [19] for each collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ the corresponding characteristic matrix $\Omega_{\tau}(\cdot)$ is given by

$$\Omega_{\tau}(\lambda) = \Omega_{\tau_0}(\lambda) - S_+(\lambda)(\tau_+(\lambda) + M_+(\lambda))^{-1}S_-^*(\bar{\lambda}), \quad \lambda \in \mathbb{C}_+ \quad (4.15)$$

This and Theorem 4.7 yield the desired statement. \square

4.3. Minimal spectral functions. We start the subsection with the following lemma which is immediate from Theorem 4.7.

Lemma 4.10. *Let $\Sigma_{\tau,\varphi} : \mathbb{R} \rightarrow [\tilde{\mathcal{K}}]$ be a spectral function of the boundary problem (2.46)-(2.48), corresponding to the solution $\varphi(t, \lambda) \in [\tilde{\mathcal{K}}, H]$ of the equation (2.29) (see Definition 4.5). Assume also that $\tilde{H} \supset \tilde{\mathcal{K}}$, $\tilde{\mathcal{K}}^\perp = \tilde{H} \ominus \tilde{\mathcal{K}}$ and $Y_W(\cdot, \lambda) \in [\tilde{H}, H]$ is a solution of (2.29) satisfying the conditions of Theorem 4.7 and the equality $Y_W(t, \lambda) \upharpoonright \tilde{\mathcal{K}} = \varphi(t, \lambda)$ (such a solution exists because $0 \in \hat{\rho}(\tilde{\varphi}(0, \lambda))$). Then the (unique) spectral function of the boundary problem (2.46)-(2.48) corresponding to $Y_W(\cdot, \lambda)$ is*

$$\Sigma_{\tau,W}(s) = \begin{pmatrix} \Sigma_{\tau,\varphi}(s) & 0 \\ 0 & 0 \end{pmatrix} : \tilde{\mathcal{K}} \oplus \tilde{\mathcal{K}}^\perp \rightarrow \tilde{\mathcal{K}} \oplus \tilde{\mathcal{K}}^\perp, \quad (4.16)$$

which implies that the spectral function $\Sigma_{\tau,\varphi}$ is unique.

Conversely if a spectral function $\Sigma_{\tau,W}$ is of the form (4.16), then $\Sigma_{\tau,\varphi}(s)$ is a spectral function corresponding to $\varphi(\cdot, \lambda)$.

Now combining Theorems 4.7, 4.8 with Lemma 4.10 and taking the equality (3.40) into account one derives the following theorem.

Theorem 4.11. *Let $N = (N_0 \ N_1)$ be an admissible operator pair (3.1) and let $\varphi_N(t, \lambda) \in [\hat{\mathcal{K}}, H]$ be the operator solution of the equation (2.29) with the initial data (3.27). Then: 1) for each collection $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ of holomorphic pairs (3.2)-(3.5) there exists a unique spectral function $\Sigma_{\mathcal{P},N} : \mathbb{R} \rightarrow [\hat{\mathcal{K}}]$ of the boundary problem (3.11)-(3.15) corresponding to $\varphi_N(\cdot, \lambda)$. This function is given by*

$$\Sigma_{\mathcal{P},N}(s) = s - \lim_{\delta \rightarrow +0} w - \lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \text{Im } m_{\mathcal{P}}(\sigma + i\varepsilon) d\sigma, \quad (4.17)$$

where $m_{\mathcal{P}}(\lambda)$ is the m -function corresponding to the boundary problem (3.11)-(3.15). Moreover, the spectral function $\Sigma_{\mathcal{P},N}$ is orthogonal if and only if $\mathcal{P} \in TR^0\{\mathcal{H}_0, \mathcal{H}_1\}$.

2) let $\Sigma_{\mathcal{P},N}(\cdot)$ be a spectral function and let $V : \mathfrak{H} \rightarrow L_2(\Sigma_{\mathcal{P},N}; \hat{\mathcal{K}})$ be an isometry given by the Fourier transform (4.5) with $\varphi(t, s) = \varphi_N(t, s)$. Then

$$(V^*g)(t) = \int_{\mathbb{R}} \varphi_N(t, s) d\Sigma_{\mathcal{P},N}(s)g(s), \quad g = g(s) \in \text{Hol}^0(\hat{\mathcal{K}}).$$

In particular the inverse Fourier transform is

$$f(t) = \int_{\mathbb{R}} \varphi_N(t, s) d\Sigma_{\mathcal{P},N}(s)g_f(s).$$

In the next theorem we give a parameterization of all spectral functions $\Sigma_{\mathcal{P},N}(\cdot)$ in terms of a boundary parameter $\mathcal{P} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$.

Theorem 4.12. *Let the assumptions of Theorem 4.11 be satisfied, let \hat{N} be the operator (3.49) and let $T_{N,0} : \mathbb{C} \setminus \mathbb{R} \rightarrow [\hat{\mathcal{K}}]$, $T_{N,+} : \mathbb{C}_+ \rightarrow [\mathcal{H}_0, \hat{\mathcal{K}}]$ and $T_{N,-} : \mathbb{C}_- \rightarrow$*

$[\mathcal{H}_1, \hat{\mathcal{K}}]$ be the operator functions defined by

$$\begin{aligned} T_{N,0}(\lambda) &= \hat{N}^* \Omega_{\tau_0}(\lambda) \hat{N}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}; \\ T_{N,+}(\lambda) &= \hat{N}^* S_+(\lambda), \quad \lambda \in \mathbb{C}_+; \quad T_{N,-}(z) = \hat{N}^* S_-(z), \quad z \in \mathbb{C}_-. \end{aligned}$$

Then for each collection $\mathcal{P} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ given by (3.2)-(3.5) the equality

$$m_{\mathcal{P}}(\lambda) = T_{N,0}(\lambda) + T_{N,+}(\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}C_1(\lambda)T_{N,-}^*(\bar{\lambda}), \quad \lambda \in \mathbb{C}_+ \quad (4.18)$$

together with (4.17) defines a (unique) spectral function $\Sigma_{\mathcal{P},N}(\cdot)$ of the boundary problem (3.11)-(3.15) corresponding to φ_N . Moreover a spectral function $\Sigma_{\mathcal{P},N}(\cdot)$ is orthogonal if and only if $\mathcal{P} \in TR^0\{\mathcal{H}_0, \mathcal{H}_1\}$.

Proof. Let $\mathcal{P} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ be defined by (3.2)-(3.5) and let $\tau_+(\lambda) \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ be the corresponding linear relation (2.13). Then by (4.15), (3.57) and the equality

$$-(\tau_+(\lambda) + M_+(\lambda))^{-1} = (C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}C_1(\lambda), \quad \lambda \in \mathbb{C}_+$$

the m -function $m_{\mathcal{P}}(\lambda)$ can be represented via (4.18). This together with Theorem 4.11 yield the required statement. \square

Next for a given collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ defined by (2.13) and (2.44), (2.45) consider the corresponding boundary problem (2.46)-(2.48). Denote by d_{\min} the minimal value of $\dim \tilde{\mathcal{K}}$ for the set of all spectral functions $\Sigma_{\tau,\varphi} : \mathbb{R} \rightarrow [\tilde{\mathcal{K}}]$ of this boundary problem (recall that according to Definition 4.5 each $\Sigma_{\tau,\varphi}$ corresponds to some operator solution $\varphi(t, \lambda) \in [\tilde{\mathcal{K}}, H]$ of the equation (2.29)).

Definition 4.13. A spectral function $\Sigma(\cdot) = \Sigma_{\tau,\varphi}(\cdot) : \mathbb{R} \rightarrow [\tilde{\mathcal{K}}]$ will be called minimal if $\dim \tilde{\mathcal{K}} = d_{\min}$.

In the following theorem we give a description of all minimal spectral functions of the "triangular" boundary problem (3.11)-(3.15).

Theorem 4.14. Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D -triplet (2.32) for L , let $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in TR\{\mathcal{H}_0, \mathcal{H}_1\}$ be a collection of holomorphic pairs (3.2)-(3.5) and let (3.11)-(3.15) be the corresponding boundary problem. Then:

1) $d_{\min} = \dim \hat{\mathcal{K}}$ and the set of all minimal spectral functions $\Sigma_{\min}(\cdot)$ is given by

$$\Sigma_{\min}(s) = X^* \Sigma_{\mathcal{P},N}(s) X, \quad (4.19)$$

where $\Sigma_{\mathcal{P},N}(s)$ is the (minimal) spectral function defined in Theorem 4.11 and X is an automorphism of the space $\hat{\mathcal{K}}$. Moreover, the minimal spectral function $\Sigma_{\min}(s)$ given by (4.19) corresponds to the operator solution $\varphi_{\min}(t, \lambda) := \varphi_N(t, \lambda) X^{-1*}$ of the equation (2.29).

2) if $\dim H = \infty$, then $d_{\min} (= \dim \hat{\mathcal{K}}) = \infty$.

Proof. 1) Let $\Sigma_{\tau,\varphi} : \mathbb{R} \rightarrow [\tilde{\mathcal{K}}]$ be a spectral function of the problem (3.11)-(3.15) corresponding to the solution $\varphi(t, \lambda) \in [\tilde{\mathcal{K}}, H]$ with $\tilde{\varphi}(0, \lambda) \equiv \tilde{\varphi}_0 \in [\tilde{\mathcal{K}}, H^{2n}]$. Since

$0 \in \hat{\rho}(\tilde{\varphi}_0)$, there are a Hilbert space $\tilde{\mathcal{K}}^\perp$ and an operator $\tilde{\psi}_0 \in [\tilde{\mathcal{K}}^\perp, H^{2n}]$ such that the operator $W = (\tilde{\varphi}_0 \ \tilde{\psi}_0)$ is an isomorphism of the space $\tilde{H} := \tilde{\mathcal{K}} \oplus \tilde{\mathcal{K}}^\perp$ onto H^{2n} .

Let $\Omega_{\tau,W}(\lambda)$ be the operator function (4.8) and let $\Sigma_{\tau,W}(\cdot)$ be the spectral function (4.7) corresponding to the solution $Y_W(\cdot, \lambda)$ (see Theorem 4.7). It follows from (2.61) that $s - \lim_{y \rightarrow \infty} \Omega_{\tau,W}(iy)/y = 0$. This and the integral representation of the Nevanlinna function $\Omega_{\tau,W}(\lambda)$ [2, 12] yield

$$\text{Ker } \text{Im } \Omega_{\tau,W}(\lambda) = \{\tilde{h} \in \tilde{H} : \Sigma_{\tau,W}(s)\tilde{h} = 0, s \in \mathbb{R}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (4.20)$$

Moreover by Lemma 4.10 the function $\Sigma_{\tau,W}(s)$ satisfies (4.16), which in view of (4.20) gives the inclusion $\tilde{\mathcal{K}}^\perp \subset \text{Ker } \text{Im } \Omega_{\tau,W}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now, letting $\tilde{H}_0 := \tilde{H} \ominus \text{Ker } \text{Im } \Omega_{\tau,W}(\lambda)$ one obtains $\dim \tilde{H}_0 \leq \dim \tilde{\mathcal{K}}$.

Next assume that $W' \in [\hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp, H^{2n}]$ is the isomorphism (3.35) and $\Omega_{\tau,W'}(\lambda)$ is the operator function (3.38). It follows from (4.8) that there exists an isomorphism $C \in [\hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp, \tilde{H}]$ such that $\Omega_{\tau,W'}(\lambda) = C^* \Omega_{\tau,W}(\lambda) C$. Moreover, by the block representation (3.40) one has $\text{Ker } \text{Im } \Omega_{\tau,W'}(\lambda) = \hat{\mathcal{K}}^\perp$. Hence $\text{Ker } \text{Im } \Omega_{\tau,W}(\lambda) = C \hat{\mathcal{K}}^\perp$ and consequently $\hat{\mathcal{K}} = C^* \tilde{H}_0$. Therefore $\dim \hat{\mathcal{K}} = \dim \tilde{H}_0 \leq \dim \tilde{\mathcal{K}}$, which yields the equality $d_{\min} = \dim \hat{\mathcal{K}}$.

To prove the relation (4.19) note that for each automorphism $X \in [\hat{\mathcal{K}}]$ this relation defines the minimal spectral function $\Sigma_{\min}(s) = \Sigma_{\tau, \varphi_{\min}}(s)$, corresponding to the solution $\varphi_{\min}(t, \lambda) := \varphi_N(t, \lambda) X^{-1*}$. Conversely, let $\Sigma_{\min}(s) = \Sigma_{\tau, \varphi_{\min}}(s)$ be a minimal spectral function corresponding to the solution $\varphi_{\min}(t, \lambda) \in [\hat{\mathcal{K}}, H]$. Since $0 \in \hat{\rho}(\tilde{\varphi}(0, \lambda)) \cap \hat{\rho}(\tilde{\varphi}_N(0, \lambda))$, there exists an automorphism $X \in [\hat{\mathcal{K}}]$ such that $\varphi_{\min}(t, \lambda) = \varphi_N(t, \lambda) X^{-1*}$ and hence the distribution $\Sigma(s) := X^* \Sigma_{\mathcal{P}, N}(s) X$ is a spectral function corresponding to φ_{\min} . Since by Lemma 4.10 such a function is unique, it follows that $\Sigma_{\min}(s) = \Sigma(s) = X^* \Sigma_{\mathcal{P}, N}(s) X$.

The statement 2) is implied by the statement 1) and the inequality (3.10) \square

Finally by using the above results we can estimate the spectral multiplicity of an exit space extension $\tilde{A} \supset L_0$. Namely, the following corollary is valid.

Corollary 4.15. *Let the assumptions of Theorem 4.14 be satisfied and let $R_{\mathcal{P}}(\lambda) = P_{\mathfrak{H}}(\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}$ be a generalized resolvent generated by the boundary problem (3.11)-(3.15). Then the spectral multiplicity of the extension \tilde{A} does not exceed $d_{\min} (= \dim \hat{\mathcal{K}})$.*

Proof. Let $\Sigma = \Sigma_{\mathcal{P}, N} : \mathbb{R} \rightarrow [\hat{\mathcal{K}}]$ be a spectral function defined in Theorem 4.11 and let $\chi'(s)$ be a bounded linear map in $\text{Hol}(\Sigma, \hat{\mathcal{K}})$ given for all $s \in \mathbb{R}$ by

$$(\chi'(s)f)(\sigma) = \chi_{(-\infty, s)}(\sigma)f(\sigma), \quad f = f(\sigma) \in \text{Hol}(\Sigma, \hat{\mathcal{K}})$$

(here $\chi_{(-\infty, s)}(\cdot)$ is the indicator of the interval $(-\infty, s)$). It is easily seen that the map $\chi'(s)$ admits the continuous extension $\chi(s) \in [L_2(\Sigma; \hat{\mathcal{K}})]$ ($s \in \mathbb{R}$) such that $\chi(\cdot)$ is an orthogonal spectral function (resolution of identity) in $L_2(\Sigma; \hat{\mathcal{K}})$.

Next assume that $V \in [\mathfrak{H}, L_2(\Sigma; \hat{\mathcal{K}})]$ is an isometry given by the Fourier transform (4.5) with $\varphi = \varphi_N$ and let $\mathcal{L} := V\mathfrak{H}$, $\tilde{\mathcal{L}} = \text{span}\{\mathcal{L}, \chi(s)\mathcal{L} : s \in \mathbb{R}\}$. As is known the subspace $\tilde{\mathcal{L}}$ reduces the spectral function $\chi(s)$ and the equality $\tilde{\chi}(s) = \chi(s) \upharpoonright \tilde{\mathcal{L}}$ defines the minimal orthogonal spectral function $\tilde{\chi}(s)$ in $\tilde{\mathcal{L}}$ (actually one can prove that $\tilde{\mathcal{L}} = L_2(\Sigma; \hat{\mathcal{K}})$). Moreover, the relation (4.6) yields

$$F_{\mathcal{P}}(t) = V^* \chi(t) V = V^* (P_{\mathcal{L}} \tilde{\chi}(t) \upharpoonright \mathcal{L}) V, \quad t \in \mathbb{R}, \quad (4.21)$$

where $F_{\mathcal{P}}(t) = P_{\mathfrak{H}} \tilde{E}(t) \upharpoonright \mathfrak{H}$ and $\tilde{E}(t)$ is the orthogonal spectral function of \tilde{A} . It follows from (4.21) that the spectral functions $F_{\mathcal{P}}(t)$ and $P_{\mathcal{L}} \tilde{\chi}(t) \upharpoonright \mathcal{L}$ are unitary equivalent and, consequently, so are the (minimal) orthogonal spectral functions $\tilde{E}(t)$ and $\tilde{\chi}(t)$. This and the fact that $\tilde{\chi}(t)$ is a part of $\chi(t)$ imply that the spectral multiplicity of $\tilde{E}(t)$ does not exceed the spectral multiplicity of $\chi(t)$, which in turn does not exceed $\dim \hat{\mathcal{K}}$. This proves the required statement. \square

Remark 4.16. It follows from Proposition 3.2 that in the case $n_{b+} < \infty$ (in particular, $\dim H < \infty$) the statements of Theorem 4.14 and Corollary 4.15 can be naturally extended to the boundary problems (2.46)-(2.48) generated by a quasi-constant Nevanlinna collection $\{C(\cdot), D(\cdot)\}$.

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DEPARTMENT OF CALCULUS, LUGANS'K NATIONAL UNIVERSITY, 2 OBORONNA, LUGANS'K,
91011, UKRAINE

E-mail address: vim@mail.dsip.net